Open Sets and Closed Sets

Open Sets

One of the themes of this (or any other) course in real analysis is the curious interplay between various notions of "big" sets and "small" sets. We have seen at least one such measure of size already: Uncountable sets are big, whereas countable sets are small. In this chapter we will make precise what was only hinted at in Chapter Three – the rather vague notion of a "thick" set in a metric space. For our purposes, a "thick" set will be one that contains an entire neighborhood of each of its points. But perhaps we can come up with a better name.... Throughout this chapter, unless otherwise specified, we live in a generic metric space (M, d).

A set U in a metric space (M, d) is called an **open set** if U contains a neighborhood of each of its points. In other words, U is an open set if, given $x \in U$, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Examples 4.1

(a) In any metric space, the whole space M is an open set. The empty set $\emptyset$ is also open (by default).

(b) In $\mathbb{R}$, any open interval is an open set. Indeed, given $x \in (a, b)$, let $\varepsilon = \min\{x - a, b - x\}$. Then, $\varepsilon > 0$ and $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. The cases (a, ∞) and $(-\infty, b)$ are similar. While we're at it, notice that the interval $[0, 1)$, for example, is *not* open in $\mathbb{R}$ because it does not contain an entire neighborhood of 0.

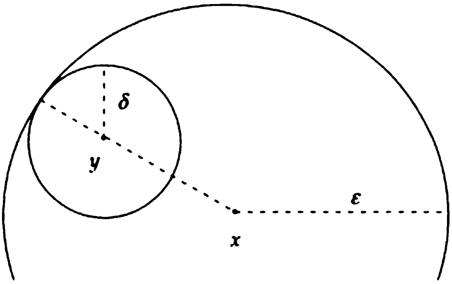
(c) In a discrete space, $B_1(x) = \{x\}$ is an open set for any x. (Why?) It follows that *every* subset of a discrete space is open.

Before we get too carried away, we should follow the lead suggested by our last two examples and check that every open ball is in fact an open set.

Proposition 4.2. *For any $x \in M$ and any $\varepsilon > 0$, the open ball $B_\varepsilon(x)$ is an open set.*

PROOF. Let $y \in B_\varepsilon(x)$. Then $d(x, y) < \varepsilon$ and hence $\delta = \varepsilon - d(x, y) > 0$. We will show that $B_\delta(y) \subset B_\varepsilon(x)$ (as in Figure 4.1). Indeed, if $d(y, z) < \delta$, then, by the triangle inequality, $d(x, z) \le d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + \varepsilon - d(x, y) = \varepsilon$. $\square$

Let's collect our thoughts. First, every open ball is open. Next, it follows from the definition of open sets that an open set must actually be a *union* of open balls. In fact,



if U is open, then $U = \bigcup\{B_\varepsilon(x) : B_\varepsilon(x) \subset U\}$. Moreover, *any* arbitrary union of open balls is again an open set. (Why?) Here's what all of this means:

Theorem 4.3. *An* arbitrary union *of open sets is again open; that is, if* $(U_\alpha)_{\alpha\in A}$ *is any collection of open sets, then* $V = \bigcup_{\alpha\in A} U_\alpha$ *is open.*

PROOF. If $x \in V$, then $x \in U_\alpha$ for some $\alpha \in A$. But then, since U_α is open, $B_\varepsilon(x) \subset U_\alpha \subset V$ for some $\varepsilon > 0$. □

Intersections aren't nearly as generous:

Theorem 4.4. *A* finite intersection *of open sets is open; that is, if each of* $U_1, \ldots, U_n$ *is open, then so is* $V = U_1 \cap \cdots \cap U_n$.

PROOF. If $x \in V$, then $x \in U_i$ for all $i = 1, \ldots, n$. Thus, for each i there is an $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(x) \subset U_i$. But then, setting $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$, we have $B_\varepsilon(x) \subset \bigcap_{i=1}^n B_{\varepsilon_i}(x) \subset \bigcap_{i=1}^n U_i = V$. □

Example 4.5

The word "finite" is crucial in Theorem 4.4 because $\bigcap_{n=1}^\infty(-1/n, 1/n) = \{0\}$, and $\{0\}$ is not open in $\mathbb{R}$. (Why?)

Now, since the real line $\mathbb{R}$ is of special interest to us, let's characterize the open subsets of $\mathbb{R}$. This will come in handy later. But it should be stressed that while this characterization holds for $\mathbb{R}$, it does not have a satisfactory analogue even in $\mathbb{R}^2$. (As we will see in Chapter Six, not every open set in the plane can be written as a union of *disjoint* open disks.)

Theorem 4.6. *If U is an open subset of $\mathbb{R}$, then U may be written as a* **countable** *union of* **disjoint** *open intervals. That is, $U = \bigcup_{n=1}^\infty I_n$, where $I_n = (a_n, b_n)$ (these may be unbounded) and $I_n \cap I_m = \emptyset$ for $n \neq m$.*

PROOF. We know that U can be written as a union of open intervals (because each $x \in U$ is in some open interval I with $I \subset U$). What we need to show is that U is a union of *disjoint* open intervals – such a union, as we know, must be countable (see Exercise 2.15).

We first claim that each $x \in U$ is contained in a *maximal* open interval $I_x \subset U$ in the sense that if $x \in I \subset U$, where I is an open interval, then we must have

$I \subset I_x$. Indeed, given $x \in U$, let

$$a_x = \inf\{a : (a, x] \subset U\} \qquad \text{and} \qquad b_x = \sup\{b : [x, b) \subset U\}.$$

Then, $I_x = (a_x, b_x)$ satisfies $x \in I_x \subset U$, and I_x is clearly maximal. (Check this!)

Next, notice that for any $x, y \in U$ we have either $I_x \cap I_y = \emptyset$ or $I_x = I_y$. Why? Because if $I_x \cap I_y \neq \emptyset$, then $I_x \cup I_y$ is an open interval containing both I_x and I_y. By maximality we would then have $I_x = I_y$. It follows that U is the union of disjoint (maximal) intervals: $U = \bigcup_{x \in U} I_x$. □

Now any time we make up a new definition in a metric space setting, it is usually very helpful to find an equivalent version stated exclusively in terms of sequences. To motivate this in the particular case of open sets, let's recall:

$$x_n \to x \iff (x_n) \text{ is eventually in } B_\varepsilon(x), \text{ for any } \varepsilon > 0$$

and hence

$$x_n \to x \iff (x_n) \text{ is eventually in } U, \text{ for any open set } U \text{ containing } x.$$

(Why?) This last statement essentially characterizes open sets:

Theorem 4.7. *A set U in (M, d) is open if and only if, whenever a sequence (x_n) in M converges to a point $x \in U$, we have $x_n \in U$ for all but finitely many n.*

PROOF. The forward implication is clear from the remarks preceding the theorem. Let's see why the new condition implies that U is open:

If U is *not* open, then there is an $x \in U$ such that $B_\varepsilon(x) \cap U^c \neq \emptyset$ for all $\varepsilon > 0$. In particular, for each n there is some $x_n \in B_{1/n}(x) \cap U^c$. But then $(x_n) \subset U^c$ and $x_n \to x$. (Why?) Thus, the new condition also fails. □

In slightly different language, Theorem 4.7 is saying that the only way to reach a member of an open set is by traveling well inside the set; there are no inhabitants on the "frontier." In essence, you cannot visit a single resident without seeing a whole neighborhood!

Closed Sets

What good would "open" be without "closed"? A set F in a metric space (M, d) is said to be a **closed set** if its complement $F^c = M \setminus F$ is open.

We can draw several immediate (although not terribly enlightening) conclusions:

Examples 4.8

(a) $\emptyset$ and M are always closed. (And so it is possible for a set to be *both* open and closed!)

(b) An arbitrary intersection of closed sets is closed. A finite union of closed sets is closed.

(c) Any finite set is closed. Indeed, it is enough to show that $\{x\}$ is always closed. (Why?) Given any $y \in M \setminus \{x\}$ (that is, any $y \neq x$), note that $\varepsilon = d(x, y) > 0$, and hence $B_\varepsilon(y) \subset M \setminus \{x\}$.

(d) In $\mathbb{R}$, each of the intervals $[a, b]$, $[a, \infty)$, and $(-\infty, b]$ is closed. Also, $\mathbb{N}$ and Δ are closed sets. (Why?)

(e) In a discrete space, *every* subset is closed.

(f) *Sets are not "doors"!* $(0, 1]$ is neither open nor closed in $\mathbb{R}$!

As yet, our definition is not terribly useful. It would be nice if we had an intrinsic characterization of closed sets – something that did not depend on a knowledge of open sets – something in terms of sequences, for example. For this let's first make an observation: F is closed if and only if F^c is open, and so F is closed if and only if

$$x \in F^c \Longrightarrow B_\varepsilon(x) \subset F^c \qquad \text{for some } \varepsilon > 0.$$

But this is the same as saying: F is closed if and only if

$$B_\varepsilon(x) \cap F \neq \emptyset \qquad \text{for every } \varepsilon > 0 \Longrightarrow x \in F. \tag{4.1}$$

This is our first characterization of closed sets. (Compare this with the phrase "F is not open," as in the proof of Theorem 4.7. They are similar, but not the same!)

Notice, please, that if $x \in F$, then $B_\varepsilon(x) \cap F \neq \emptyset$ necessarily follows; we are interested in the reverse implication here. In general, a point x that satisfies $B_\varepsilon(x) \cap F \neq \emptyset$ for every $\varepsilon > 0$ is evidently "very close" to F in the sense that x cannot be separated from F by any positive distance. At worst, x might be on the "boundary" of F. Thus condition (4.1) is telling us that a set is closed if and only if it contains all such "boundary" points. Exercises 33, 40, and 41 make these notions more precise. For now, let's translate condition (4.1) into a sequential characterization of closed sets.

Theorem 4.9. *Given a set F in (M, d), the following are equivalent:*

(i) *F is closed; that is, $F^c = M \setminus F$ is open.*

(ii) *If $B_\varepsilon(x) \cap F \neq \emptyset$ for every $\varepsilon > 0$, then $x \in F$.*

(iii) *If a sequence $(x_n) \subset F$ converges to some point $x \in M$, then $x \in F$.*

PROOF. (i) $\Longleftrightarrow$ (ii): This is clear from our observations above and the definition of an open set.

(ii) $\Longrightarrow$ (iii): Suppose that $(x_n) \subset F$ and $x_n \xrightarrow{d} x \in M$. Then $B_\varepsilon(x)$ contains infinitely many x_n for any $\varepsilon > 0$, and hence $B_\varepsilon(x) \cap F \neq \emptyset$ for any $\varepsilon > 0$. Thus $x \in F$, by (ii).

(iii) $\Longrightarrow$ (ii): If $B_\varepsilon(x) \cap F \neq \emptyset$ for all $\varepsilon > 0$, then for each n there is an $x_n \in B_{1/n}(x) \cap F$. The sequence (x_n) satisfies $(x_n) \subset F$ and $x_n \to x$. Hence, by (iii), $x \in F$. □

Condition (iii) of Theorem 4.9 is just a rewording of our sequential characterization of open sets (Theorem 4.7) applied to $U = F^c$. Most authors take (iii) as the definition of a closed set. In other words, condition (iii) says that a closed set must contain all of its *limit points*. That is, "closed" means closed under the operation of taking of limits. (Exercise 33 explores a slightly different, but more precise, notion of limit point.)

EXERCISES

1. Show that an "open rectangle" $(a, b) \times (c, d)$ is an open set in $\mathbb{R}^2$. More generally, if A and B are open in $\mathbb{R}$, show that $A \times B$ is open in $\mathbb{R}^2$. If A and B are closed in $\mathbb{R}$, show that $A \times B$ is closed in $\mathbb{R}^2$.

2. If F is a closed set and G is an open set in a metric space M, show that $F \setminus G$ is closed and that $G \setminus F$ is open.

▷ 3. Some authors say that two metrics d and ρ on a set M are equivalent if they generate the same open sets. Prove this. (Recall that we have defined equivalence to mean that d and ρ generate the same convergent sequences. See Exercise 3.42.)

4. Prove that *every* subset of a metric space M can be written as the intersection of open sets.

▷ 5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$ and that $\{x : f(x) = 0\}$ is a closed subset of $\mathbb{R}$.

6. Give an example of an infinite closed set in $\mathbb{R}$ containing only irrationals. Is there an open set consisting entirely of irrationals?

7. Show that every open set in $\mathbb{R}$ is the union of (countably many) open intervals with *rational* endpoints. Use this to show that the collection $\mathcal{U}$ of all open subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$ itself.

▷ 8. Show that every open interval (and hence every open set) in $\mathbb{R}$ is a countable union of closed intervals and that every closed interval in $\mathbb{R}$ is a countable intersection of open intervals.

9. Let d be a metric on an infinite set M. Prove that there is an open set U in M such that both U and its complement are infinite. [Hint: Either (M, d) is discrete or it's not....]

10. Given $y = (y_n) \in H^\infty$, $N \in \mathbb{N}$, and $\varepsilon > 0$, show that $\{x = (x_n) \in H^\infty : |x_k - y_k| < \varepsilon,\ k = 1, \ldots, N\}$ is open in H^∞ (see Exercise 3.10).

▷ 11. Let $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$, where the kth entry is 1 and the rest are 0s. Show that $\{e^{(k)} : k \geq 1\}$ is closed as a subset of ℓ_1.

12. Let F be the set of all $x \in \ell_\infty$ such that $x_n = 0$ for all but finitely many n. Is F closed? open? neither? Explain.

13. Show that c_0 is a closed subset of ℓ_∞. [Hint: If $(x^{(n)})$ is a sequence (of sequences!) in c_0 converging to $x \in \ell_\infty$, note that $|x_k| \leq \left|x_k - x_k^{(n)}\right| + \left|x_k^{(n)}\right|$ and now choose n so that $\left|x_k - x_k^{(n)}\right|$ is small *independent* of k.]

14. Show that the set $A = \{x \in \ell_2 : |x_n| \leq 1/n,\ n = 1, 2, \ldots\}$ is a closed set in ℓ_2 but that $B = \{x \in \ell_2 : |x_n| < 1/n,\ n = 1, 2, \ldots\}$ is *not* an open set. [Hint: Does $B \supset B_\varepsilon(0)$?]

Now, as we've seen, some sets are neither open nor closed. However, it is possible to describe the "open part" of a set and the "closure" of a set. Here's what we'll do:

Given a set A in (M, d), we define the **interior** of A, written $\text{int}(A)$ or A°, to be the largest open set contained in A. That is,

$$\begin{aligned}\text{int}(A) = A^\circ &= \bigcup\{U : U \text{ is open and } U \subset A\} \\ &= \bigcup\{B_\varepsilon(x) : B_\varepsilon(x) \subset A \text{ for some } x \in A,\ \varepsilon > 0\} \qquad \text{(Why?)} \\ &= \{x \in A : B_\varepsilon(x) \subset A \text{ for some } \varepsilon > 0\}.\end{aligned}$$

Note that A° is clearly an *open subset* of A.

We next define the **closure** of A, written $\text{cl}(A)$ or $\bar{A}$, to be the smallest closed set containing A. That is,

$$\text{cl}(A) = \bar{A} = \bigcap\{F : F \text{ is closed and } A \subset F\}.$$

Please take note of the "dual" nature of our two new definitions.

Now it is clear that $\bar{A}$ is a *closed* set containing A – and necessarily the smallest one. But it's not so clear which points are in $\bar{A}$ or, more precisely, which points are in $\bar{A} \setminus A$. We could use a description of $\bar{A}$ that is a little easier to "test" on a given set A. It follows from our last theorem that $x \in \bar{A}$ if and only if $B_\varepsilon(x) \cap \bar{A} \neq \emptyset$ for every $\varepsilon > 0$. The description that we are looking for simply removes this last reference to $\bar{A}$.

Proposition 4.10. *$x \in \bar{A}$ if and only if $B_\varepsilon(x) \cap A \neq \emptyset$ for every $\varepsilon > 0$.*

PROOF. One direction is easy: If $B_\varepsilon(x) \cap A \neq \emptyset$ for every $\varepsilon > 0$, then $B_\varepsilon(x) \cap \bar{A} \neq \emptyset$ for every $\varepsilon > 0$, and hence $x \in \bar{A}$ by Theorem 4.9.

Now, for the other direction, let $x \in \bar{A}$ and let $\varepsilon > 0$. If $B_\varepsilon(x) \cap A = \emptyset$, then A is a subset of $(B_\varepsilon(x))^c$, a closed set. Thus, $\bar{A} \subset (B_\varepsilon(x))^c$. (Why?) But this is a contradiction, because $x \in \bar{A}$ while $x \notin (B_\varepsilon(x))^c$. □

Corollary 4.11. *$x \in \bar{A}$ if and only if there is a sequence $(x_n) \subset A$ with $x_n \to x$.*

That is, $\bar{A}$ is the set of all limits of convergent sequences in A (including limits of constant sequences).

Example 4.12

Here are a few easy examples in $\mathbb{R}$. (Check the details!)

(a) $\text{int}((0, 1]) = (0, 1)$ and $\text{cl}((0, 1]) = [0, 1]$,

(b) $\text{int}(\{(1/n) : n \geq 1\}) = \emptyset$ and $\text{cl}(\{(1/n) : n \geq 1\}) = \{(1/n) : n \geq 1\} \cup \{0\}$,

(c) $\text{int}(\mathbb{Q}) = \emptyset$ and $\text{cl}(\mathbb{Q}) = \mathbb{R}$,

(d) $\text{int}(\Delta) = \emptyset$ and $\text{cl}(\Delta) = \Delta$.

EXERCISES

Unless otherwise specified, each of the following exercises is set in a generic metric space (M, d).

15. The set $A = \{y \in M : d(x, y) \le r\}$ is sometimes called the *closed ball* about x of radius r. Show that A is a closed set, but give an example showing that A need not equal the closure of the open ball $B_r(x)$.

16. If $(V, \|\cdot\|)$ is any normed space, prove that the closed ball $\{x \in V : \|x\| \le 1\}$ is always the closure of the open ball $\{x \in V : \|x\| < 1\}$.

▷ **17.** Show that A is open if and only if $A^\circ = A$ and that A is closed if and only if $\bar{A} = A$.

▷ **18.** Given a nonempty bounded subset E of $\mathbb{R}$, show that $\sup E$ and $\inf E$ are elements of $\bar{E}$. Thus $\sup E$ and $\inf E$ are elements of E whenever E is *closed*.

▷ **19.** Show that $\operatorname{diam}(A) = \operatorname{diam}(\bar{A})$.

20. If $A \subset B$, show that $\bar{A} \subset \bar{B}$. Does $\bar{A} \subset \bar{B}$ imply $A \subset B$? Explain.

21. If A and B are any sets in M, show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. Give an example showing that this last inclusion can be proper.

22. True or false? $(A \cup B)^\circ = A^\circ \cup B^\circ$.

23. If $x \ne y$ in M, show that there are disjoint open sets U, V with $x \in U$ and $y \in V$. Moreover, show that U and V can be chosen so that even $\bar{U}$ and $\bar{V}$ are disjoint.

24. Show that $\bar{A} = (\operatorname{int}(A^c))^c$ and that $A^\circ = (\operatorname{cl}(A^c))^c$.

25. A set that is simultaneously open and closed is sometimes called a **clopen** set. Show that $\mathbb{R}$ has no nontrivial clopen sets. [Hint: If U is a nontrivial open subset of $\mathbb{R}$, show that $\bar{U}$ is strictly bigger than U.]

26. We define the distance from a point $x \in M$ to a nonempty set A in M by $d(x, A) = \inf\{d(x, a) : a \in A\}$. Prove that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

27. Show that $|d(x, A) - d(y, A)| \le d(x, y)$ and conclude that the map $x \mapsto d(x, A)$ is continuous.

28. Given a set A in M and $\varepsilon > 0$, show that $\{x \in M : d(x, A) < \varepsilon\}$ is an open set and that $\{x \in M : d(x, A) \le \varepsilon\}$ is a closed set (and each contains A).

29. Show that every closed set in M is the intersection of countably many open sets and that every open set in M is the union of countably many closed sets. [Hint: What is $\bigcap_{n=1}^{\infty}\{x \in M : d(x, A) < (1/n)\}$?]

30.

(a) For each $n \in \mathbb{Z}$, let F_n be a closed subset of $(n, n+1)$. Show that $F = \bigcup_{n\in\mathbb{Z}} F_n$ is a closed set in $\mathbb{R}$. [Hint: For each fixed n, first show that there is a $\delta_n > 0$ so that $|x - y| \ge \delta_n$ whenever $x \in F_n$ and $y \in F_m$, $m \ne n$.]

(b) Find a sequence of disjoint closed sets in $\mathbb{R}$ whose union is *not* closed.

31. If $x \notin F$, where F is closed, show that there are disjoint open sets U, V with $x \in U$ and $F \subset V$. (This extends the first result in Exercise 23 since $\{y\}$ is closed.) Is it possible to find U and V so that $\bar{U}$ and $\bar{V}$ are disjoint? Is it possible to extend this result further to read: Any two disjoint closed sets are contained in disjoint open sets?

32. We define the distance between two (nonempty) subsets A and B of M by $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Give an example of two disjoint closed sets A and B in $\mathbb{R}^2$ with $d(A, B) = 0$.

▷ **33.** Let A be a subset of M. A point $x \in M$ is called a **limit point** of A if every neighborhood of x contains a point of A that is different from x itself, that is, if $(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ for every $\varepsilon > 0$. If x is a limit point of A, show that every neighborhood of x contains infinitely many points of A.

▷ **34.** Show that x is a limit point of A if and only if there is a sequence (x_n) in A such that $x_n \to x$ and $x_n \neq x$ for all n.

35. Let A' be the set of limit points of a set A. Show that A' is closed and that $\bar{A} = A' \cup A$. Show that $A' \subset A$ if and only if A is closed. (A' is called the *derived set* of A.)

36. Suppose that $x_n \xrightarrow{d} x \in M$, and let $A = \{x\} \cup \{x_n : n \geq 1\}$. Prove that A is closed.

37. Prove the Bolzano–Weierstrass theorem: Every bounded infinite subset of $\mathbb{R}$ has a limit point. [Hint: Use the nested interval theorem. If A is a bounded infinite subset of $\mathbb{R}$, then A is contained in some closed bounded interval I_1. At least one of the left or right halves of I_1 contains infinitely many points of A. Call this new closed interval I_2. Continue.]

38. A set P is called **perfect** if it is empty or if it is a closed set and every point of P is a limit point of P. Show that Δ is perfect. Show that $\mathbb{R}$ is perfect when considered as a subset of $\mathbb{R}^2$.

39. Show that a nonempty perfect subset P of $\mathbb{R}$ is uncountable. This gives yet another proof that the Cantor set is uncountable. [Hint: First convince yourself that P is infinite, and assume that P is countable, say $P = \{x_1, x_2, \ldots\}$. Construct a decreasing sequence of nested closed intervals $[\, a_n, b_n \,]$ such that $(a_n, b_n) \cap P \neq \emptyset$ but $x_n \notin [\, a_n, b_n \,]$. Use the nested interval theorem to get a contradiction.]

40. If $x \in A$ and x is *not* a limit point of A, then x is called an **isolated point** of A. Show that a point $x \in A$ is an isolated point of A if and only if $(B_\varepsilon(x) \setminus \{x\}) \cap A = \emptyset$ for some $\varepsilon > 0$. Prove that a subset of $\mathbb{R}$ can have at most countably many isolated points, thus showing that every uncountable subset of $\mathbb{R}$ has a limit point.

41. Related to the notion of limit points and isolated points are boundary points. A point $x \in M$ is said to be a **boundary point** of A if each neighborhood of x hits both A and A^c. In symbols, x is a boundary point of A if and only if $B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$ for every $\varepsilon > 0$. Verify each of the following formulas, where $\text{bdry}(A)$ denotes the set of boundary points of A:
(a) $\text{bdry}(A) = \text{bdry}(A^c)$,
(b) $\text{cl}(A) = \text{bdry}(A) \cup \text{int}(A)$,
(c) $M = \text{int}(A) \cup \text{bdry}(A) \cup \text{int}(A^c)$.
Notice that the first and last equations tell us that each set A partitions M into three regions: the points "well inside" A, the points "well outside" A, and the points on the common boundary of A and A^c.

42. If E is a nonempty bounded subset of $\mathbb{R}$, show that $\sup E$ and $\inf E$ are both boundary points of E. Hence, if E is also closed, then $\sup E$ and $\inf E$ are elements of E.

43. Show that bdry(A) is always a closed set; in fact, bdry$(A) = \bar{A} \setminus A^\circ$.

44. Show that A is closed if and only if bdry$(A) \subset A$.

45. Give examples showing that bdry$(A) = \emptyset$ and bdry$(A) = M$ are both possible.

▷ **46.** A set A is said to be **dense** in M (or, as some authors say, *everywhere dense*) if $\bar{A} = M$. For example, both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$. Show that A is dense in M if and only if any of the following hold:

(a) Every point in M is the limit of a sequence from A.

(b) $B_\varepsilon(x) \cap A \neq \emptyset$ for every $x \in M$ and every $\varepsilon > 0$.

(c) $U \cap A \neq \emptyset$ for every nonempty open set U.

(d) A^c has empty interior.

47. Let G be open and let D be dense in M. Show that $\overline{G \cap D} = \bar{G}$. Give an example showing that this equality may fail if G is not open.

▷ **48.** A metric space is called **separable** if it contains a countable dense subset. Find examples of countable dense sets in $\mathbb{R}$, in $\mathbb{R}^2$, and in $\mathbb{R}^n$.

49. Prove that ℓ_2 and H^∞ are separable. [Hint: Consider finitely nonzero sequences of the form $(r_1, \ldots, r_n, 0, 0, \ldots)$, where each r_k is rational.]

50. Show that ℓ_∞ is *not* separable. [Hint: Consider the set $2^{\mathbb{N}}$, consisting of all sequences of 0s and 1s, as a subset of ℓ_∞. We know that $2^{\mathbb{N}}$ is uncountable. Now what?]

51. Show that a separable metric space has at most countably many isolated points.

52. If M is separable, show that any collection of disjoint open sets in M is at most countable.

53. Can you find a countable dense subset of $C[0, 1]$?

54. A set A is said to be **nowhere dense** in M if int (cl(A)) $= \emptyset$. Show that $\{x\}$ is nowhere dense if and only if x is *not* an isolated point of M.

55. Show that every finite subset of $\mathbb{R}$ is nowhere dense. Is every countable subset of $\mathbb{R}$ nowhere dense? Show that the Cantor set is nowhere dense in $\mathbb{R}$.

56. If A and B are nowhere dense in M, show that $A \cup B$ is nowhere dense. Give an example showing that an infinite union of nowhere dense sets need not be nowhere dense.

57. If A is closed, show that A is nowhere dense if and only if A^c is dense if and only if A has an empty interior.

58. Let (r_n) be an enumeration of $\mathbb{Q}$. For each n, let I_n be the open interval centered at r_n of radius 2^{-n}, and let $U = \bigcup_{n=1}^\infty I_n$. Prove that U is a proper, open, dense subset of $\mathbb{R}$ and that U^c is nowhere dense in $\mathbb{R}$.

59. If A is closed, show that bdry(A) is nowhere dense.

60. Show that each of the following is equivalent to the statement "A is nowhere dense":

(a) $\bar{A}$ contains no nonempty open set.

(b) Each nonempty open set in M contains a nonempty open subset that is disjoint from A.

(c) Each nonempty open set in M contains an open ball that is disjoint from A.

The Relative Metric

Although it is a digression at this point, we need to generate some terminology for later use. First, given a nontrivial subset A of a metric space (M, d), recall that A "inherits" the metric d by restriction. Thus, the metric space (A, d) has open sets, closed sets, convergent sequences, and so on, of its own. How are these related to the open sets, closed sets, convergent sequences, and so on, of (M, d)? The answer comes from examining the open balls in (A, d). Note that for $x \in A$ we have

$$B_\varepsilon^A(x) = \{a \in A : d(x, a) < \varepsilon\} = A \cap \{y \in M : d(x, y) < \varepsilon\} = A \cap B_\varepsilon^M(x),$$

where superscripts have been used to distinguish between a ball in A and a ball in M. Thus, a subset G of A is open in (A, d), or open *relative* to A, if, given $x \in G$, there is some $\varepsilon > 0$ such that

$$G \supset B_\varepsilon^A(x) = A \cap B_\varepsilon^M(x).$$

This observation leads us to the following:

Proposition 4.13. *Let $A \subset M$.*

(i) *A set $G \subset A$ is open in (A, d) if and only if $G = A \cap U$, where U is open in (M, d).*

(ii) *A set $F \subset A$ is closed in (A, d) if and only if $F = A \cap C$, where C is closed in (M, d).*

(iii) $\text{cl}_A(E) = A \cap \text{cl}_M(E)$ *for any subset E of A (where the subscripts distinguish between the closure of E in (A, d) and the closure of E in (M, d)).*

PROOF. We will prove (i) and leave the rest as exercises.

First suppose that $G = A \cap U$, where U is open in (M, d). If $x \in G \subset U$, then $x \in B_\varepsilon^M(x) \subset U$ for some $\varepsilon > 0$. But since $G \subset A$, we have $x \in A \cap B_\varepsilon^M(x) = B_\varepsilon^A(x) \subset A \cap U = G$. Thus, G is open in (A, d).

Next suppose that G is open in (A, d). Then, for each $x \in G$, there is some $\varepsilon_x > 0$ such that $x \in B_{\varepsilon_x}^A(x) = A \cap B_{\varepsilon_x}^M(x) \subset G$. But now it is clear that $U = \bigcup\{B_{\varepsilon_x}^M(x) : x \in G\}$ is an open set in (M, d) satisfying $G = A \cap U$. □

We paraphrase the statement "G is open in (A, d)" by saying that "G is open in A," or "G is open relative to A," or perhaps "G is relatively open in A." The same goes for closed sets. In the case of closures, the symbols $\mathrm{cl}_A(E)$ are read "the closure of E in A." Another notation for $\mathrm{cl}_A(E)$ is $\bar{E}^A$.

Examples 4.14

(a) Let $A = (0, 1] \cup \{2\}$, considered as a subset of $\mathbb{R}$. Then, $(0, 1]$ is open in A and $\{2\}$ is both open and closed in A. (Why?)

(b) We may consider $\mathbb{R}$ as a subset of $\mathbb{R}^2$ in an obvious way – all pairs of the form $(x, 0)$, $x \in \mathbb{R}$. The metric that $\mathbb{R}$ inherits from $\mathbb{R}^2$ in this way is nothing but the usual metric on $\mathbb{R}$. (Why?) Similarly, $\mathbb{R}^2$ may be considered as a natural subset of $\mathbb{R}^3$ (as the xy-plane, for instance). What happens in this case? Figure 4.2 might help.

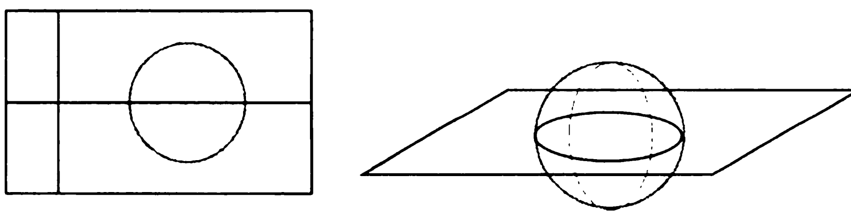


Figure 4.2

EXERCISES

Throughout, M denotes an arbitrary metric space with metric d.

▷ **61.** Complete the proof of Proposition 4.13.

▷ **62.** Suppose that A is open in (M, d) and that $G \subset A$. Show that G is open in A if and only if G is open in M. Is the result still true if "open" is replaced everywhere by "closed"? Explain.

63. Is there a nonempty subset of $\mathbb{R}$ that is open when considered as a subset of $\mathbb{R}^2$? closed?

64. Show that the analogue of part (iii) of Proposition 4.13 for relative interiors is *false*. Specifically, find sets $E \subset A \subset \mathbb{R}$ such that $\mathrm{int}_A(E) = A$ while $\mathrm{int}_{\mathbb{R}}(E) = \emptyset$.

65. Let A be a subset of M. If G and H are disjoint open sets in A, show that there are disjoint open sets U and V in M such that $G = U \cap A$ and $H = V \cap A$. [Hint: Let $U = \bigcup\{B^M_{\varepsilon/2}(x) : x \in G \text{ and } B^A_\varepsilon(x) \subset G\}$. Do the same for V and H.]

66. Let $A \subset B \subset M$. If A is dense in B (how would you define this?), and if B is dense in M, show that A is dense in M.

67. Let G be open and let D be dense in M. Show that $G \cap D$ is dense in G. Give an example showing that this may fail if G is not open.

68. If A is a separable subset of M (that is, if A has a countable dense subset of its own), show that $\bar{A}$ is also separable.

69. A collection (U_α) of open sets is called an *open base* for M if every open set in M can be written as a union of U_α. For example, the collection of all open intervals in $\mathbb{R}$ with *rational* endpoints is an open base for $\mathbb{R}$ (and this is even a countable collection). (Why?) Prove that M has a countable open base if and only if M is separable. [Hint: If $\{x_n\}$ is a countable dense set in M, consider the collection of open balls with rational radii centered at the x_n.]

◊

Notes and Remarks

For sets of real numbers, the concepts of neighborhoods, limit points (Exercise 33), derived sets (Exercise 35), perfect sets (Exercise 38), closed sets, and the characterization of open sets (Theorem 4.6) are all due to Cantor. Fréchet introduced separable spaces (Exercise 48). Much of the terminology that we use today is based on that used by either Fréchet or Hausdorff. For more details on the history of these notions see Dudley [1989], Manheim [1964], Taylor [1982], and Willard [1970]; also see Fréchet [1928], Haussdorf [1937], and Hobson [1927].

For an alternate proof of Theorem 4.6, see Labarre [1965], and for more on "Cantor-like" nowhere dense subsets of $\mathbb{R}$ (as in Exercise 58), see the short note in Wilansky [1953b].