HYPERSPACES OF KELLER COMPACTA AND THEIR ORBIT SPACES

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Abstract. A compact convex subset K of a topological linear space is called a Keller compactum if it is affinely homeomorphic to an infinite-dimensional compact convex subset of the Hilbert space ℓ_2. Let G be a compact topological group acting affinely on a Keller compactum K and let 2^K denote the hyperspace of all non-empty compact subsets of K endowed with the Hausdorff metric topology and the induced action of G. Further, let $cc(K)$ denote the subspace of 2^K consisting of all compact convex subsets of K. In a particular case, the main result of the paper asserts that if K is centrally symmetric, then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube.

1. Introduction

By a *Keller compactum* we mean a compact convex subset of a topological linear space that is affinely homeomorphic to an infinite-dimensional compact convex subset of the real separable Hilbert space ℓ_2 (see [8, Ch. III, § 3]). It is well known that every infinite-dimensional compact convex subset of an arbitrary Fréchet space (in particular, of a Banach space) is a Keller compactum (see [8, Ch. III, §3, Proposition 3.1]).

The Hilbert cube Q is the simplest but most important example of a Keller compactum. It is the compact convex subset $[-1,1]^\infty = \prod_{n=1}^{\infty}[-1,1]_n$ of the Fréchet space $\mathbb{R}^\infty$, whose product topology is induced by the following standard metric:

$$\rho(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|, \quad x = (x_n),\, y = (y_n) \in Q.$$

It is well-known that the Hilbert cube Q is affinely homeomorphic to the compact convex subset

$$H = \{x \in \ell_2 \mid |x_n| \leq 1/n,\, n \in \mathbb{N}\}$$

of the Hilbert space ℓ_2, known under the name of a *fundamental parallelopiped* of ℓ_2 or *the Hilbert brick* (see [13, Ch. 3, §11]).

In 1931, O. H. Keller [11] proved that every infinite-dimensional compact convex set in ℓ_2 is homeomorphic to the Hilbert cube Q, and in 1955, V. L. Klee Jr. [12] extended this result to arbitrary normed linear spaces. Nevertheless, not all Keller compacta are affinely homeomorphic to each other [8, Ch. V, §4]. Thus, besides the topological properties of Q, a Keller compactum carries an affine-topological structure. Furthermore, a continuous action of a topological group on a Keller compactum K broadens the structure of K to a geometric-topological one. In this sense, we study the affine-topological structure induced by a Keller compactum K in the hyperspaces 2^K and $cc(K)$ (which are defined below), as well as the topological structure of certain orbit spaces of the latter ones.

Throughout the paper K will denote a Keller compactum. As usual, 2^K denotes the hyperspace of all non-empty compact subsets of K endowed with the topology induced

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by the Hausdorff metric:

$$d_H(A,B)=\max\left\{\sup_{b\in B} d(b,A),\quad \sup_{a\in A} d(a,B)\right\},\quad A,B\in 2^K,$$

where d is any compatible metric on K. Recall that all compatible metrics on K induce the same topology on 2^K [16, Theorem 3.2].

By $cc(K)$ we denote the subspace of 2^K consisting of all compact convex subsets of K.

Let G be a compact topological group acting affinely on K, i.e., every $g\in G$ satisfies the equality (2.2) below. In this paper we study the induced action of G on the hyperspaces 2^K and $cc(K)$. Namely, the action of G on K induces a continuous action of G on 2^K, which is given by the rule:

$$(g,A)\mapsto gA=\{ga\mid a\in A\},\quad g\in G,\ \ A\in 2^K. \tag{1.1}$$

Continuity of this action easily follows. Indeed, consider the map $r: G\times 2^K\to 2^{G\times K}$ given by the rule $r\big((g,A)\big)=\{g\}\times A$. The action defined in the formula (1.1) can now be written as the composition $\theta=2^F\circ r$ where $F: G\times K\to K$ is the action of G on K. Since r and 2^F are continuous, the composition is also continuous.

Obviously, the hyperspace $cc(K)$ is an invariant subset of 2^K under the action (1.1).

By Curtis-Schori-West Hyperspace Theorem (see, e.g., [14, Theorem 8.4.5]), 2^K is homeomorphic to the Hilbert cube. It was proved in [15] that for any compact convex subset X of a locally convex metrizable linear space with $\dim X>1$, the hyperspace $cc(X)$ is homeomorphic to the Hilbert cube. Since K is affinely homeomorphic to an infinite-dimensional compact convex subset V of ℓ_2, $cc(K)$ is homeomorphic to $cc(V)$ (see [16, Theorem 1.3]). Consequently, these two facts yield that $cc(K)$ is homeomorphic to the Hilbert cube.

Our interest in orbit spaces of hyperspaces of Keller compacta relies on the relationship between such classical objects like the Banach-Mazur compacta $BM(n)$, $n\geq 2$, from one hand and the orbit spaces of certain geometrically defined hyperspaces of the Euclidean closed unit ball $\mathbb{B}^n=\{(x_1,\dots,x_n)\in\mathbb{R}^n\mid \sum_{i=1}^n x_i^2\leq 1\}$, from the other hand (see [4]). Since the Keller compacta K are natural infinite-dimensional analogs of $\mathbb{B}^n$, studying the topological structure of orbit spaces of 2^K and $cc(K)$ with respect to compact topological groups acting affinely on K seems quite natural and interesting.

The main goal of this paper is to prove that if there exists a G-fixed point in the radial interior of K then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube (see Theorem 4.1). In Corollary 4.4 we show that if K has a center of symmetry, then the latter one is fixed under every element $g\in G$. Since the Hilbert cube Q satisfies this condition, we get the homeomorphisms $2^Q/G\cong Q$ and $cc(Q)/G\cong Q$ (see Corollary 4.5). As a by-product, we give also a short and easy proof of the above mentioned homeomorphism $cc(K)\cong Q$ for Keller compacta K with non-empty radial interior.

2. Preliminaries

We refer the reader to the monographs [9] and [17] for basic notions of the theory of G-spaces. However, we recall here some special definitions and results that will be used throughout the paper.

All maps between topological spaces are assumed to be continuous. A map $f: X\to Y$ between G-spaces is called G-*equivariant* (or simply *equivariant*) if $f(gx)=gf(x)$ for every $x\in X$ and $g\in G$.

Let (X,d) be a metric G-space. If $d(gx,gy)=d(x,y)$ for every $x,y\in X$ and $g\in G$, then we say that d is a G-*invariant* (or simply *invariant*) metric. That is, every $g\in G$ is actually an isometry of X with respect to the metric d.

Let G be a compact group and X a metric G-space with an invariant metric d. It is well-known (see, e.g., [17, Proposition 1.1.12]) that the quotient topology of the orbit space X/G is generated by the metric

$$d^*(G(x), G(y)) = \inf_{g\in G} d(x, gy), \quad G(x), G(y) \in X/G.$$

Evidently,

$$d^*(G(x), G(y)) \leq d(x, y), \quad x, y \in X. \tag{2.1}$$

For a given topological group G, a metrizable G-space X is called a *G-equivariant absolute neighborhood retract* (denoted by $X \in G$-ANR) if for any metrizable G-space Z containing X as an invariant closed subset, there exist an invariant neighborhood U of X in Z and a G-retraction $r : U \to X$. If we can always take $U = Z$, then we say that X is a *G-equivariant absolute retract* (denoted by $X \in G$-AR).

A point x_0 in a G-space X is called a *G-fixed point* if $gx_0 = x_0$ for every $g \in G$.

A *Hilbert cube manifold* or a *Q-manifold* is a separable metrizable space that admits an open cover each member of which is homeomorphic to an open subset of the Hilbert cube Q. We refer the reader to [10] and [14] for the theory of Q-manifolds.

Let V and V' be convex subsets of linear spaces L and L' respectively. A map $f : V \to V'$ is called *affine*, if for any $n \geq 1$, one has

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i f(x_i), \tag{2.2}$$

whenever $x_i \in V$, $t_i \geq 0$ and $\sum_{i=1}^{n} t_i = 1$.

A point $x_0 \in K$ is said to be *internal* [8, Definition 4.2] if for every $x \in K$,

$$\inf\{|t| \mid x_0 + t(x - x_0) \notin K\} > 0.$$

Equivalently, $x_0 \in K$ is internal if for every $x \in K$, there exists $t < 0$ such that $x_0 + t(x - x_0)$ belongs to K (see [7, p. 162]).

The set of all internal points of K is called the *radial interior* of K and is denoted by $\operatorname{rint} K$. The complement $K \setminus \operatorname{rint} K$ is called the *radial boundary* of K and is denoted by $\operatorname{rbd} K$.

Whereas the radial boundary of any Keller compactum is a dense subset (see [8, Ch. V, §4, Corollary 4.2]), there exist Keller compacta with empty radial interior. An example of this is given in [8, p. 161]. However, if $\operatorname{rint} K \neq \emptyset$, then

$$\operatorname{rint} K = x_0 + [0, 1)(K - x_0) \tag{2.3}$$

for every $x_0 \in \operatorname{rint} K$ and it is also a dense subset of K [8, Ch. V, §4, Proposition 4.4].

Observe that the definition of $\operatorname{rint} K$ has been stated in affine topological terms. In particular, this notion is invariant under affine homeomorphism, i.e., if $\xi : K \to V$ is an affine homeomorphism of two Keller compacta, then $\xi(\operatorname{rint} K) = \operatorname{rint} V$, or equivalently, $\xi(\operatorname{rbd} K) = \operatorname{rbd} V$.

It is clear that if a Keller compactum K has a center of symmetry, then it must belong to $\operatorname{rint} K$.

Recall that a point $x_0 \in K$ is a *center of symmetry* if for every $x \in K$, there exists a $y \in K$ such that $x_0 = (x + y)/2$. If K admits a center of symmetry, then it is called *centrally symmetric*. It is well-known and easy to see that any centrally symmetric compact convex subset of a normed linear space has exactly one center of symmetry; we shall use this fact in the proof of Corollary 4.4 below.

The convex hull of a subset $A \subset K$ will be denoted by $\operatorname{conv} A$. Let d be a compatible metric on K. For any $r > 0$ and $A \in 2^K$, we denote by

$$B(A, r) = \{x \in K \mid d(x, A) < r\} \quad \text{and} \quad C(A, r) = \{x \in K \mid d(x, A) \leq r\},$$

the open r-neighborhood and the closed r-neighborhood of A in K, respectively.

Let G be a topological group and L a real topological linear space. We call L a *linear G-space* if it is endowed with a linear action of G, i.e., if

$$g(\lambda x + y) = \lambda(gx) + gy$$

for every $g \in G$, $\lambda \in \mathbb{R}$ and $x, y \in L$. If, in addition, L admits an invariant norm $\|\cdot\| : L \to \mathbb{R}$, i.e.,

$$\|gx\| = \|x\| \tag{2.4}$$

for every $g \in G$ and $x \in L$, then we call L a *normed linear G-space*. In this case, the metric induced by the norm is invariant:

$$\|gx - gy\| = \|g(x-y)\| = \|x-y\|. \tag{2.5}$$

If moreover $(L, \|\cdot\|)$ is a Banach space, then we call L a *Banach G-space*.

For a compact topological group G, we denote by $C(G, L)$ the real topological linear space of all maps from G to L endowed with the compact-open topology. The action of G on $C(G, L)$ is given by the rule:

$$(gf)(x) = f(xg), \quad g, x \in G, \quad f \in C(G, L), \tag{2.6}$$

(see [2, Proposition 4]). This action turns $C(G, L)$ into a linear G-space. Furthermore, if $(L, \|\cdot\|)$ is a Banach space, then the supremum norm on $C(G, L)$:

$$\|f\| = \sup_{x \in G} \|f(x)\|, \tag{2.7}$$

is invariant. Indeed,

$$\|gf\| = \sup_{x \in G} \|f(xg)\| = \sup_{y \in G} \|f(y)\| = \|f\|.$$

Thus, $C(G, L)$ becomes a Banach G-space. Observe that due to compactness of G, the topology induced by the norm (2.7) on $C(G, L)$ is just the compact-open one.

Recall that a metric space X is called *continuum-connected* if each pair of points in X is contained in a subcontinuum. X is *locally continuum-connected* if it has an open base of continuum-connected subsets.

The following theorems will play an essential role in our proofs.

Theorem 2.1 ([3, Theorem 8]). *Let G be a compact group and X a separable G-*ANR *(resp., a G-*AR*). Then the orbit space X/G is an* ANR *(resp., an* AR*).*

Theorem 2.2 ([5, Proposition 3.1]). *Let G be a compact group and X a locally continuum-connected (resp., connected and locally continuum-connected) metrizable G-space. Then 2^X is a G-*ANR *(resp., a G-*AR*).*

3. Equivariant affine embedding in a Banach G-space

In this section we prove the following equivariant embedding result.

Proposition 3.1. *Let G be a compact group acting affinely on a Keller compactum K. Then there is an affine equivariant embedding of K into a Banach G-space.*

Proof. Let $h : K \to \ell_2$ be an affine embedding. Then, h induces an equivariant embedding $\widetilde{h} : K \to C(G, \ell_2)$ according to the rule:

$$\widetilde{h}(x)(g) = h(gx), \quad x \in K, \quad g \in G, \tag{3.1}$$

where $C(G, \ell_2)$ is endowed with the linear action of G defined by the formula (2.6) (see [20, Theorem 2], cf. [2, Theorem 3]). Moreover, since ℓ_2 is a Banach space, the supremum norm on $C(G, \ell_2)$ (see formula (2.7)) turns $C(G, \ell_2)$ into a Banach G-space. Next, since G acts affinely on K and h is an affine map, $\widetilde{h}$ is also an affine map. Indeed, let $n \in \mathbb{N}$, $x_i \in K$ and $t_i \geq 0$ such that $\sum_{i=1}^n t_i = 1$. Then for every $g \in G$ we have

$$\begin{aligned}\widetilde{h}\Big(\sum_{i=1}^n t_i x_i\Big)(g) &= h\Big(g \sum_{i=1}^n t_i x_i\Big) = h\Big(\sum_{i=1}^n t_i g x_i\Big) = \sum_{i=1}^n t_i h(g x_i) \\ &= \sum_{i=1}^n t_i \Big(\widetilde{h}(x_i)(g)\Big) = \sum_{i=1}^n \Big(t_i \widetilde{h}(x_i)\Big)(g) = \Big(\sum_{i=1}^n t_i \widetilde{h}(x_i)\Big)(g).\end{aligned}$$

Hence,

$$\widetilde{h}\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i \widetilde{h}(x_i),$$

showing that $\widetilde{h}$ is an affine map. Thus, K embedds as an invariant convex subset in the Banach G-space $C(G, \ell_2)$. □

Corollary 3.2. *Let G be a compact group acting affinely on a Keller compactum K. Then $K \in G$-*AR *and consequently, the orbit space $K/G \in$* AR.

Proof. That $K \in G$-AR follows directly from Proposition 3.1 and [1, Theorem 2] (see also [2, Corollary 7]). Then, Theorem 2.1 implies that $K/G \in$ AR. □

Proposition 3.3. *Let G be a compact group that acts affinely on Keller compacta K and V and assume that $\xi : K \to V$ is an affine G-equivariant homeomorphism. Then the induced hyperspace map $2^\xi : \big(2^K, cc(K)\big) \to \big(2^V, cc(V)\big)$ is a G-equivariant homeomorphism of the pairs yielding the homeomorphy of the respective G-orbit spaces. Furthermore, if there is a G-fixed pont $x_0 \in \operatorname{rint} K$, then $\xi(x_0)$ is a G-fixed point belonging to* $\operatorname{rint} V$.

Proof. By [16, Theorem 1.3], the hyperspace map $2^\xi : 2^K \to 2^V$, which is defined by $2^\xi(A) = \xi(A)$, is a homeomorphism. Since ξ is an affine map, 2^ξ restricts to a homeomorphism $2^\xi\,|_{cc(K)}$ from $cc(K)$ onto $cc(V)$. Furthermore, the G-equivariance of ξ implies the G-equivariance of 2^ξ and $2^\xi\,|_{cc(K)}$. Next, since x_0 is a G-fixed point and ξ is equivariant, $\xi(x_0)$ is also a G-fixed point. Finally, since the radial interior is invariant of an affine homeomorpism, $x_0 \in \operatorname{rint} K$ implies that $\xi(x_0) \in \operatorname{rint} V$. □

4. Orbit spaces of 2^K and $cc(K)$

In this section we prove the main result of the paper:

Theorem 4.1. *Let G be a compact group acting affinely on a Keller compactum K. If K has a G-fixed point $x_0 \in \operatorname{rint} K$, then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube.*

We begin with the following proposition.

Proposition 4.2. *Let G be a compact group acting affinely on a Keller compactum K. Then the orbit space $cc(K)/G$ is a compact* AR.

Proof. Since the notions involved are affine-topological, we may assume that $K \subset \ell_2$. By [19, Lemma 2.1], the closed convex hull operator $\overline{\operatorname{conv}} : 2^K \to cc(K)$; $A \mapsto \overline{\operatorname{conv} A}$, is a (continuous) retraction. Since every $g \in G$ preserves convex combinations, this retraction is an equivariant map. By Theorem 2.2, 2^K is a compact G-AR. Hence, $cc(K)$, being an equivariant retract of 2^K, is also a compact G-AR. Therefore, by Theorem 2.1, the orbit space $cc(K)/H$ is a compact AR. □

Lemma 4.3. *If there is a G-fixed point $x_0 \in \operatorname{rint} K$, then for every $\varepsilon > 0$, there exist G-equivariant maps $\varphi, \psi : \big(2^K, cc(K)\big) \to \big(2^V, cc(V)\big)$, ϵ-close to the identity map of 2^K such that $\operatorname{Im}\varphi \cap \operatorname{Im}\psi = \emptyset$.*

Proof. Since K is a compact convex set, there exists $0 < \lambda < 1$ such that

$$d(x, x_0 + \lambda(x - x_0)) < \epsilon/2 \tag{4.1}$$

for every $x \in K$.

Let $\varphi : 2^K \to 2^K$ be defined by

$$\varphi(A) = x_0 + \lambda(A - x_0) = \{x_0 + \lambda(a - x_0) \mid a \in A\}, \quad A \in 2^K.$$

Then φ is continuous, since it is just the hyperspace map 2^f of the map $f : K \to K$ defined by $f(x) = x_0 + \lambda(x - x_0)$ (see [14, §5.3]). Since G acts affinely on K and x_0 is a G-fixed point, the map φ is G-equivariant, for if $h \in G$ and $A \in 2^K$, then

$$\begin{aligned}\varphi(hA) &= \{x_0 + \lambda(ha - x_0) \mid a \in A\} = \{h(x_0 + \lambda(a - x_0)) \mid a \in A\}\\ &= h\{x_0 + \lambda(a - x_0) \mid a \in A\} = h\varphi(A).\end{aligned}$$

To see that φ is ϵ-close to the identity map of 2^K, take $A \in 2^K$. Then

$$d(a, \varphi(A)) \le d(a, x_0 + \lambda(a - x_0)) \quad \text{and} \quad d(x_0 + \lambda(a - x_0), A) \le d(x_0 + \lambda(a - x_0), a)$$

for every $a \in A$. Consequently, by inequality (4.1), $d_H(A, \varphi(A)) \le \epsilon/2 < \epsilon$.

Note that by equality (2.3), $\varphi(A) \subset \operatorname{rint} K$. This yields that $\varphi(A) \cap \operatorname{rbd} K = \emptyset$ for every $A \in 2^K$.

Next, let $\psi : 2^K \to 2^K$ be defined by

$$\psi(A) = \{x \in C \mid d(x, A) \le \epsilon/2\}, \quad A \in 2^C.$$

Then $\psi(A)$ is just the closed $\epsilon/2$-neighborhood of A in K. Continuity of ψ is a well known fact. Indeed, it follows from the inequality $d_H(\psi(A), \psi(B)) \le d_H(A, B)$ for all $A, B \in 2^K$, if we take into account that d, being induced by a norm, is a geodesic (or convex) metric (see [16, Proposition 10.5]).

The G-equivariance of ψ follows from the G-invariance of d (see the equalities (2.4) and (2.5)). Clearly, ψ is ϵ-close to the identity map of 2^K. Finally, since $\operatorname{rbd} K$ is dense in K (see [8, Ch. V, §4, Corollary 4.2]), we infer that $\psi(A) \cap \operatorname{rbd} K \ne \emptyset$ for every $A \in 2^K$. Therefore, $\operatorname{Im}\varphi \cap \operatorname{Im}\psi = \emptyset$, as required.

It remains to observe that $\varphi(A) \in cc(K)$ whenever $A \in cc(K)$. On the other hand, since the metric in K is induced by a norm, the compact set $\psi(A)$ is also convex for every $A \in cc(K)$. This completes the proof. □

Proof of Theorem 4.1. By Proposition 3.1 and 3.3, we can assume that K is an invariant subset of a Banach G-space and K admits a G-fixed point $x_0 \in \operatorname{rint} K$.

First we consider the case of $2^K/G$. It follows from Theorems 2.1 and 2.2 that the orbit space $2^K/G$ is a compact AR. Thus, by [10, Theorem 22.1], it remains to show that $2^K/G$ is a Q-manifold. According to Toruńczyk's Characterization Theorem (see [22]), it suffices to show that there exist maps $f_1, f_2 : 2^K/G \to 2^K/G$, arbitrarily close to the identity map of $2^K/G$ such that $\operatorname{Im} f_1 \cap \operatorname{Im} f_2 = \emptyset$.

Let $\epsilon > 0$. By Lemma 4.3, there exist G-equivariant maps $\varphi, \psi : 2^K \to 2^K$, ϵ-close to the identity map of 2^K with $\operatorname{Im}\varphi \cap \operatorname{Im}\psi = \emptyset$. Let $\widetilde{\varphi} : 2^K/G \to 2^K/G$ and $\widetilde{\psi} : 2^K/G \to 2^K/G$ be the maps induced by φ and ψ, respectively (see [17, Proposition 1.1.17]). By inequality (2.1), the maps $\widetilde{\varphi}$ and $\widetilde{\psi}$ are ϵ-close to the identity map of $2^K/G$. Finally, $\widetilde{\varphi}$ and $\widetilde{\psi}$ have disjoint images, since $\operatorname{Im}\varphi \cap \operatorname{Im}\psi = \emptyset$ and

$$\operatorname{Im}\widetilde{\varphi} \cap \operatorname{Im}\widetilde{\psi} = \frac{\operatorname{Im}\varphi}{G} \cap \frac{\operatorname{Im}\psi}{G} = \frac{\operatorname{Im}\varphi \cap \operatorname{Im}\psi}{G}.$$

This completes the proof for $2^K/G$.

For $cc(K)/G$ the argument is quite analogous. Indeed, by Proposition 4.2, the orbit space is a compact AR. Then the restrictions $\widetilde{\varphi}|_{cc(K)/G}$ and $\widetilde{\psi}|_{cc(K)/G}$ are ε-close to the identity map of $cc(K)/G$ and have disjoint images. The proof of Theorem 4.1 is now complete. □

Corollary 4.4. *Let K be any centrally symmetric Keller compactum. Then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube.*

Proof. Let $y_0 \in K$ be the (unique) center of symmetry of K. Hence, for every $x \in K$, there is a point $y \in K$ such that $y_0 = (x + y)/2$. Then y_0 belongs to the segment $[y, x] \subset K$, and thus, $y_0 \in \operatorname{rint} K$. Uniqueness of the center of symmetry implies that y_0 is a G-fixed point, for if there is an $h \in G$ such that $y_0 \ne hy_0$, then there is a point $x \in K$

such that for every $y \in K$, $hy_0 \neq (x+y)/2$. Let $z \in K$ be such that $y_0 = (h^{-1}x+z)/2$. Since G acts affinely on K, we have that $hy_0 = (x+hz)/2$, a contradiction. The result now follows from Theorem 4.1. □

Let G be a compact group acting affinely on the Hilbert cube Q. Then the origin of $\mathbb{R}^\infty$ is the center of symmetry of Q, and hence, it is a G-fixed point. Due to a particular importance of this case, we state it as a separate corollary.

Corollary 4.5. *Let G be a compact group acting affinely on the Hilbert cube Q. Then the orbit spaces $2^Q/G$ and $cc(Q)/G$ are homeomorphic to the Hilbert cube.*

In connection with Theorem 4.1 we have the following remark.

Remark 4.6. *In case G is a compact Lie group acting non-transitively on a non-degenerate Peano continuum X (i.e., a locally connected compact metrizable space that contains more than one point), the following stronger result of the first author* [6] *is worth mentioning: the orbit space $2^X/G$ is homeomorphic to the Hilbert cube.*

However, since there exist compact non-Lie groups acting affinely on Keller compacta, the above Theorem 4.1 is not a particular case of this result. For instance, the closed subgroup of the group $\mathrm{Iso}(Q)$ *of isometries of the Hilbert cube (Q,ρ) consisting of all isometries $g : Q \to Q$ such that $g(x)_n = \pm x_n$ for every $x = (x_n) \in Q$, is topologically isomorphic to the Cantor group $\mathbb{Z}_2^\infty$, which is not a Lie group.*

5. Concluding remarks and questions

Typically, compact groups that act effectively and affinely on the Hilbert cube Q are the groups of affine isometries and their closed subgroups. More precisely, let $\mathcal{H}(Q)$ denote the topological group of all homeomorphisms of the Hilbert cube Q equipped with the compact open topology. Denote by $\mathrm{Aff}\,(Q)$ the closed subgroup of $\mathcal{H}(Q)$ consisting of all affine homeomorphisms. Further, for every compatible metric d on Q, the group $\mathrm{Iso}\,(Q,d)$ of all d-isometries of Q is a compact subgroup of $\mathcal{H}(Q)$ (by the way, the compact open topology on $\mathrm{Iso}\,(Q,d)$ coincides with the topology of pointwise convergence). The intersection $\mathrm{Afis}\,(Q) = \mathrm{Aff}\,(Q) \cap \mathrm{Iso}\,(Q,d)$ is just the group of affine d-isometries of Q.

Clearly, $\mathrm{Afis}\,(Q)$ is a compact group which acts effectively on Q via the evaluation map, i.e., $g * x = g(x)$ for every $g \in \mathrm{Afis}\,(Q)$ and $x \in Q$. Moreover, any compact group G which acts effectively and affinely on Q is a subgroup (up to a topological isomorphism) of $\mathrm{Afis}\,(Q)$ for some compatible metric d. Indeed, parting from any compatible metric σ of Q, define $d_\sigma(x,y) = \sup\limits_{g\in G} \sigma(gx,gy)$. Then d_σ is a compatible G-invariant metric for Q. Next, the map $G \to \mathrm{Iso}\,(Q,d_\sigma)$ which sends an element $g \in G$ to the isomorphism $\tilde{g} \in \mathrm{Iso}\,(Q,d_\sigma)$ defined by $\tilde{g}(x) = gx$, $x \in Q$, is a topological monomorphism of topological groups. Therefore, if in addition, G acts affinely on Q, then it is topologically isomorphic to a subgroup of $\mathrm{Afis}\,(Q,d_\sigma)$.

It is worth mentioning that $\mathcal{H}(Q)$ is homeomorphic to the Hilbert space ℓ_2 (it was established in Torunczyk [21] that $\mathcal{H}(Q)$ is an ℓ_2-manifold wile the contractibility of $\mathcal{H}(Q)$ was proved earlier in Renz [18]). However, there are no results concerning topological structures of the subgroups of $\mathcal{H}(Q)$ above mentioned. More specifically, we ask the following.

Question 5.1. *Describe topological structures of the topological groups* $\mathrm{Aff}\,(Q)$, $\mathrm{Iso}\,(Q,\rho)$ *and* $\mathrm{Iso}\,(H,d)$, *where ρ is the standart metric on the Hilbert cube Q and d is the ℓ_2-metric on the Hilbert brick H. In particular, are these groups absolute neighborhood retracts?*

We end the paper with the following question.

Question 5.2. *Is Theorem 4.1 still true for Keller compacta with empty radial interior and infinite G?*

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