AFFINE GROUP ACTING ON HYPERSPACES OF COMPACT CONVEX SUBSETS OF $\mathbb{R}^n$

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Abstract. For every $n \geq 2$, let $cc(\mathbb{R}^n)$ denote the hyperspace of all nonempty compact convex subsets of the Euclidean space $\mathbb{R}^n$ endowed with the Hausdorff metric topology. Let $cb(\mathbb{R}^n)$ be the subset of $cc(\mathbb{R}^n)$ consisting of all compact convex bodies. In this paper we discover several fundamental properties of the natural action of the affine group $\mathrm{Aff}(n)$ on $cb(\mathbb{R}^n)$. We prove that the space $E(n)$ of all n-dimensional ellipsoids is an $\mathrm{Aff}(n)$-equivariant retract of $cb(\mathbb{R}^n)$. This is applied to show that $cb(\mathbb{R}^n)$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$, where Q stands for the Hilbert cube. Furthermore, we investigate the action of the orthogonal group $O(n)$ on $cc(\mathbb{R}^n)$. In particular, we show that if $K \subset O(n)$ is a closed subgroup that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, then the orbit space $cc(\mathbb{R}^n)/K$ is homeomorphic to the Hilbert cube with a removed point, while $cb(\mathbb{R}^n)/K$ is a contractible Q-manifold homeomorphic to the product $(E(n)/K) \times Q$. The orbit space $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ is homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$, while $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to the open cone over $\mathrm{BM}(n)$.

1. Introduction

Let $cc(\mathbb{R}^n)$ denote the hyperspace of all nonempty compact subsets of the Euclidean space $\mathbb{R}^n$, $n \geq 1$, equipped with the Hausdorff metric:

$$d_H(A, B) = \max \left\{ \sup_{b \in B} d(b, A),\ \sup_{a \in A} d(a, B) \right\},$$

where d is the standard Euclidean metric on $\mathbb{R}^n$.

By $cb(\mathbb{R}^n)$ we shall denote the subspace of $cc(\mathbb{R}^n)$ consisting of all compact convex bodies of $\mathbb{R}^n$, i.e.,

$$cb(\mathbb{R}^n) = \{A \in cc(\mathbb{R}^n) \mid \mathrm{Int}\, A \neq \emptyset\}.$$

It is easy to see that $cc(\mathbb{R}^1)$ is homeomorphic to the closed semi-plane $\{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$, while $cb(\mathbb{R}^1)$ is homeomorphic to $\mathbb{R}^2$. In [21] it was proved that for $n \geq 2$, $cc(\mathbb{R}^n)$ is homeomorphic to the punctured Hilbert cube, i.e., Hilbert cube with a removed point. Furthermore, a simple combination of [6, Corollary 8] and [7, Theorem 1.4] yields that the hyperspace

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$\mathcal{B}(n)$, consisting of all centrally symmetric (about the origin) convex bodies $A \in cb(\mathbb{R}^n)$, $n \geq 2$, is homeomorphic to $\mathbb{R}^p \times Q$, where Q denotes the Hilbert cube and $p = n(n+1)/2$. However, the topological structure of $cb(\mathbb{R}^n)$ remained open.

In this paper we study the topological structure of the hyperspace $cb(\mathbb{R}^n)$. Namely, we will show that $cb(\mathbb{R}^n)$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$. Our argument is based on some fundamental properties of the natural action of the affine group $\mathrm{Aff}(n)$ on $cb(\mathbb{R}^n)$. On this way we prove that $\mathrm{Aff}(n)$ acts properly on $cb(\mathbb{R}^n)$ (Theorem 3.3). Using a well-known result in affine convex geometry about the minimal-volume ellipsoid, we construct a convenient global $O(n)$-slice $L(n)$ for $cb(\mathbb{R}^n)$. Namely, as it was proved by F. John [17], for each $A \in cb(\mathbb{R}^n)$ there exists a unique minimal-volume ellipsoid $l(A)$ that contains A (see also [15]). It turns out that the map $l : cb(\mathbb{R}^n) \to E(n)$ is an $\mathrm{Aff}(n)$-equivariant retraction onto the subset $E(n)$ of $cb(\mathbb{R}^n)$ consisting of all n-dimensional ellipsoids (see Theorem 3.6). Then the convenient global $O(n)$-slice of $cb(\mathbb{R}^n)$ is just the inverse image $L(n) = l^{-1}(\mathbb{B}^n)$ of the n-dimensional closed Euclidean unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. In other words, $L(n)$ is the subspace of $cb(\mathbb{R}^n)$ consisting of all convex bodies A for which $\mathbb{B}^n$ is the minimal-volume ellipsoid. This fact yields that the two orbit spaces $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ and $L(n)/O(n)$ are homeomorphic (Corollary 3.7(2)). Taking into account the compactness of $L(n)$ (see Proposition 3.4(d)) we rediscover Macbeath's result [20] from early fifties to the effect that $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ is compact (Corollary 3.7(1)).

We show in Corollary 3.9 that $cb(\mathbb{R}^n)$ is homeomorphic (even $O(n)$-equivariantly) to the product $L(n) \times E(n)$. Further, in Section 5 we prove that $L(n)$ is homeomorphic to the Hilbert cube (Corollary 5.9), while $E(n)$ is homeomorphic to $\mathbb{R}^{n(n+3)/2}$ (see Corollary 3.10). Thus, we get that $cb(\mathbb{R}^n)$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$ (Corollary 3.11), one of the main results of the paper.

In Corollary 3.8 we prove that the orbit space $cb(\mathbb{R}^n)/\mathrm{Aff}(n)$ is homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$. Recall that $\mathrm{BM}(n)$ is the set of isometry classes of n-dimensional Banach spaces topologized by the following metric best known in Functional Analysis as the Banach-Mazur distance:

$$d(E,F) = \ln \inf \big\{ \|T\| \cdot \|T^{-1}\| \;\big|\; T : E \to F \text{ is a linear isomorphism} \big\}.$$

These spaces were introduced in 1932 by S. Banach [11] and they continue to be of interest. The original geometric representation of $\mathrm{BM}(n)$ is based on the one-to-one correspondence between norms and odd symmetric convex bodies (see [30, p. 644] and [19, p. 1191]). A. Pelczyński's question of whether the Banach-Mazur compacta $\mathrm{BM}(n)$ are homeomorphic to the Hilbert cube (see [30, Problem 899]) was answered negatively for $n = 2$ by the first author [6]; the case $n \geq 3$ still remains open. The reader can find other results concerning the Banach-Mazur compacta and related spaces in [7].

In Section 4 we study the hyperspace $M(n)$ of all compact convex subsets of the unit ball $\mathbb{B}^n$ which intersect the boundary sphere $\mathbb{S}^{n-1}$. It is established in Corollary 4.13 that for every closed subgroup $K \subset O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, the K-orbit space $M(n)/K$ is homeomorphic to the Hilbert cube. In particular, $M(n)$ is homeomorphic to the Hilbert cube. On the other hand, $M_0(n)/K$ is a Hilbert cube manifold for each closed subgroup $K \subset O(n)$, where $M_0(n) = M(n) \setminus \{\mathbb{B}^n\}$. In Theorem 4.16 it is established that the orbit space $M(n)/O(n)$ is just homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$. The main technique we develop in this section is further applied to Section 5 as well. Here we establish analogous properties of the global $O(n)$-slice $L(n)$ of the proper $\mathrm{Aff}(n)$-space $cb(\mathbb{R}^n)$ (see Proposition 5.8, Corollary 5.9 and Theorem 5.11).

In Sections 6 and 7 we investigate some orbit spaces of $cc(\mathbb{R}^n)$ and $cb(\mathbb{R}^n)$. We prove in Theorem 7.1 that if K is a closed subgroup of $O(n)$ which acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, then the orbit space $cc(\mathbb{R}^n)/K$ is homeomorphic to the punctured Hilbert cube. The orbit space $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to the open cone over the Banach-Mazur compactum $\mathrm{BM}(n)$ (Theorem 7.2). Respectively, the orbit space $cb(\mathbb{R}^n)/K$ is a contractible Q-manifold homeomorphic to the product $(E(n)/K) \times Q$ (see Theorem 6.1), while topological structure of the orbit space $cb(\mathbb{R}^n)/O(n)$ mainly remains unknown.

The paper consists of the following 7 sections:

§1. Introduction.
§2. Preliminaries.
§3. Affine group acting properly on $cb(\mathbb{R}^n)$.
§4. The hyperspace $M(n)$.
§5. Some properties of $L(n)$.
§6. Orbit spaces of $cb(\mathbb{R}^n)$.
§7. Orbit spaces of $cc(\mathbb{R}^n)$.

2. Preliminaries

We refer the reader to the monographs [12] and [22] for basic notions of the theory of G-spaces. However we will recall here some special definitions and results which will be used throughout the paper.

All topological spaces and topological groups are assumed to be Tychonoff.

If G is a topological group and X is a G-space, for any $x \in X$ we denote by G_x the stabilizer of x, i.e., $G_x = \{g \in G \mid gx = x\}$. For a subset $S \subset X$ and a subgroup $H \subset G$, $H(S)$ denotes the H-saturation of S, i.e., $H(S) = \{hs \mid h \in H, s \in S\}$. If $H(S) = S$ then we say that S is an H-invariant set. In particular, $G(x)$ denotes the G-orbit of x, i.e., $G(x) = \{gx \in X \mid g \in G\}$. The orbit space is denoted by X/G.

For each subgroup $H \subset G$, the H-fixed point set X^H is the set $\{x \in X \mid H \subset G_x\}$. Clearly, X^H is a closed subset of X.

The family of all subgroups of G that are conjugate to H is denoted by $[H]$, i.e., $[H] = \{gHg^{-1} \mid g \in G\}$. We will call $[H]$ a G-orbit type (or simply an orbit type). For two orbit types $[H_1]$ and $[H_2]$, one says that $[H_1] \preceq [H_2]$ iff $H_1 \subset gH_2g^{-1}$ for some $g \in G$. The relation $\preceq$ is a partial ordering on the set of all orbit types. Since $G_{gx} = gG_xg^{-1}$ for any $x \in X$ and $g \in G$, we have $[G_x] = \{G_{gx} \mid g \in G\}$.

A continuous map $f : X \to Y$ between two G-spaces is called *equivariant* or a *G-map* if $f(gx) = g(fx)$ for every $x \in X$ and $g \in G$. If the action of G on Y is trivial and $f : X \to Y$ is an equivariant map, then we will say that f is an *invariant* map.

For any subgroup $H \subset G$, we will denote by G/H the G-space of cosets $\{gH \mid g \in G\}$ equipped with the action induced by left translations.

A G-space X is called *proper* (in the sense of Palais [23]) if it has an open cover consisting of, so called, *small* sets. A set $S \subset X$ is called small if any point $x \in X$ has a neighborhood V such that the set $\langle S, V\rangle = \{g \in G \mid gS \cap V \neq \emptyset\}$, called the transporter from S to V, has compact closure in G.

Each orbit in a proper G-space is closed, and each stabilizer is compact ([23, Proposition 1.1.4]). If G is a locally compact group and Y is a proper G-space, then for every point $y \in Y$ the orbit $G(y)$ is G-homeomorphic to G/G_y [23, Proposition 1.1.5].

For a given topological group G, a metrizable G-space Y is called a *G-equivariant absolute neighborhood retract* (denoted by $Y \in G$-ANR) if for any metrizable G-space M containing Y as an invariant closed subset, there exist an invariant neighborhood U of Y in M and a G-retraction $r : U \to Y$. If we can always take $U = M$, then we say Y is a *G-equivariant absolute retract* (denoted by $Y \in G$-AR).

Let us recall the well known definition of a slice [23, p. 305]:

Definition 2.1. Let X be a G-space and H a closed subgroup of G. An H-invariant subset $S \subset X$ is called an H-slice in X, if $G(S)$ is open in X and there exists a G-equivariant map $f : G(S) \to G/H$ such that $S{=}f^{-1}(eH)$. The saturation $G(S)$ is called a *tubular* set. If $G(S) = X$, then we say that S is *a global* H-slice of X.

In case of a compact group G one has the following intrinsic characterization of H-slices. A subset $S \subset X$ of a G-space X is an H-slice if and only if it satisfies the following four conditions: (1) S is H-invariant, (2) $G(S)$ is open in X, (3) S is closed in $G(S)$, (4) if $g \in G \setminus H$ then $gS \cap S = \emptyset$ (see [12, Ch. II, §4 and §5]).

The following is one of the fundamental results in the theory of topological transformation groups (see, e.g., [12, Ch. II, §4 and §5]):

Theorem 2.2 (Slice Theorem)**.** Let G be a compact Lie group, X a Tychonoff G-space and $x \in X$ any point. Then:

(1) There exists a G_x-slice $S \subset X$ such that $x \in S$.
(2) $[G_y] \preceq [G_x]$ for each point $y \in G(S)$.

Let G be a compact Lie group and X a G-space. By a *G-normal cover* of X, we mean a family

$$\mathcal{U} = \{gS_\mu \mid g \in G,\ \mu \in M\}$$

where each S_μ is an H_μ-slice for some closed subgroup H_μ of G, the family of saturations $\{G(S_\mu)\}_{\mu\in M}$ is an open cover for X and there exists a locally finite invariant partition of unity $\{p_\mu : X \to [0,1] \mid \mu \in M\}$ subordinated to $\{G(S_\mu)\}_{\mu\in M}$. That is to say, each p_μ is an invariant function with $\overline{p_\mu^{-1}\big((0,1]\big)} \subset G(S_\mu)$ and the supports $\{\overline{p_\mu^{-1}\big((0,1]\big)} \mid \mu \in M\}$ constitute a locally finite family. We refer to [7] for further information concerning G-normal covers.

Yet another result which plays an important role in the paper is the following one:

Theorem 2.3 (Orbit Space Theorem [4])**.** Let G be a compact Lie group and X a G-ANR (resp., a G-AR). Then the orbit space X/G is an ANR (resp., an AR).

Let (X,d) be a metric G-space. If $d(gx,gy)=d(x,y)$ for every $x,y\in X$ and $g\in G$, then we will say that d is a G-invariant (or simply invariant) metric.

Suppose that G is a compact group acting on a metric space (X,d). If d is G-invariant, it is well-known [22, Proposition 1.1.12] that the quotient topology of X/G is generated by the metric

$$d^*(G(x),G(y))=\inf_{g\in G} d(x,gy),\quad G(x),G(y)\in X/G. \tag{2.1}$$

It is evident that

$$d^*(G(x),G(y))\leq d(x,y),\quad x,y\in X. \tag{2.2}$$

In the sequel we will denote by d the Euclidean metric on $\mathbb{R}^n$. For any $A\subset\mathbb{R}^n$, and $\varepsilon>0$, we denote $N(A,\varepsilon)=\{x\in\mathbb{R}^n\mid d(x,A)<\varepsilon\}$. In particular, for every $x\in\mathbb{R}^n$, $N(x,\varepsilon)$ denotes the open ε-ball around x. On the other hand, if $\mathcal{C}\subset cc(\mathbb{R}^n)$ then for every $A\in\mathcal{C}$ we shall use $O(A,\varepsilon)$ for the ε-open ball centered at the point A in $\mathcal{C}$, i.e.,

$$O(A,\varepsilon)=\{B\in\mathcal{C}\mid d_H(A,B)<\varepsilon\},$$

where d_H stands for the Hausdorff metric induced by d.

For every subset $A\subset X$ of a topological space X, we will use the symbols ∂A and $\overline{A}$ to denote, respectively, the boundary and the closure of A in X.

We will denote by $\mathbb{B}^n$ the n-dimensional Euclidean closed unit ball and by $\mathbb{S}^{n-1}$ the corresponding unit sphere, i.e.,

$$\mathbb{B}^n=\Big\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\Big|\; \sum_{i_1}^{n} x_i^2\leq 1\Big\}\qquad\text{and}$$

$$\mathbb{S}^{n-1}=\Big\{(x_1,\dots,x_n)\in\mathbb{R}^n \;\Big|\; \sum_{i_1}^{n} x_i^2= 1\Big\}.$$

The Hilbert cube $[0,1]^\infty$ will be denoted by Q. By $cc(\mathbb{B}^n)$ we denote the subspace of $cc(\mathbb{R}^n)$ consisting of all $A\in cc(\mathbb{R}^n)$ such that $A\subset\mathbb{B}^n$. It is well known that $cc(\mathbb{B}^n)$ is homeomorphic to the Hilbert cube (see [21, Theorem 2.2]).

A Hilbert cube manifold or a Q-manifold is a separable, metrizable space that admits an open cover each member of which is homeomorphic to an open subset of the Hilbert cube Q. We refer to [14] and [28] for the theory of Q-manifolds.

A closed subset A of a metric space (X,d) is called a Z-set if the set $\{f\in C(Q,X)\mid f(Q)\cap A=\emptyset\}$ is dense in $C(Q,X)$, being $C(Q,X)$ the space of all continuous maps from Q to X endowed with the compact-open topology. In particular, if for every $\varepsilon > 0$ there exists a map $f : X \to X \setminus A$ such that $d(x, f(x)) < \varepsilon$, then A is a Z-set.

A map $f : X \to Y$ between topological spaces is called proper provided that $f^{-1}(C)$ is compact for each compact set $C \subset Y$. A proper map $f : X \to Y$ between ANR's is called cell-like (abbreviated CE) if it is onto and each point inverse $f^{-1}(y)$ has the property UV^∞. That is to say, for each neighborhood U of $f^{-1}(y)$ there exists a neighborhood $V \subset U$ of $f^{-1}(y)$ such that the inclusion $V \hookrightarrow U$ is homotopic to a constant map of V into U. In particular, if $f^{-1}(y)$ is contractible, then it has the property UV^∞ (see [14, Ch. XIII]).

3. Affine group acting properly on $cb(\mathbb{R}^n)$

Let (X, d) be a metric space and G a topological group acting continuously on X. Consider the hyperspace 2^X consisting of all nonempty compact subsets of X equipped with the Hausdorff metric topology. Define an action of G on 2^X by the rule:

$$(g, A) \longmapsto gA, \quad gA = \{ga \mid a \in A\}. \tag{3.1}$$

The reader can easily verify the continuity of this action.

3.1. **Properness of the** $\mathrm{Aff}(n)$**-action on** $cb(\mathbb{R}^n)$**.** Throughout the paper, n will always denote a natural number greater than or equal to 2.

We will denote by $\mathrm{Aff}(n)$ the group of all affine transformations of $\mathbb{R}^n$. Let us recall the definition of $\mathrm{Aff}(n)$. For every $v \in \mathbb{R}^n$ let $T_v : \mathbb{R}^n \to \mathbb{R}^n$ be the translation by v, i.e., $T_v(x) = v + x$ for all $x \in \mathbb{R}^n$. The set of all such translations is a group isomorphic to the additive group of $\mathbb{R}^n$. For every $\sigma \in GL(n)$ and $v \in \mathbb{R}^n$ it is easy to see that $\sigma T_v \sigma^{-1} = T_{\sigma(v)}$. This yields a homomorphism from $GL(n)$ to the group of all linear automorphisms of $\mathbb{R}^n$, and hence, we have an (internal) semidirect product:

$$\mathbb{R}^n \rtimes GL(n)$$

called the *Affine Group* of $\mathbb{R}^n$ (see e.g. [2, p. 102]). Each element $g \in \mathrm{Aff}(n)$ is usually represented by $g = T_v + \sigma$, where $\sigma \in GL(n)$ and $v \in \mathbb{R}^n$, i.e.,

$$g(x) = v + \sigma(x), \text{ for every } x \in \mathbb{R}^n,$$

As a semidirect product, $\mathrm{Aff}(n)$ is topologized by the product topology of $\mathbb{R}^n \times GL(n)$ thus becoming a Lie group with two connected components. Since the topology of $GL(n)$ is the one inherited from $\mathbb{R}^{n^2}$, we can also give a natural topological embedding of $\mathrm{Aff}(n)$ into $\mathbb{R}^n \times \mathbb{R}^{n^2} = \mathbb{R}^{n(n+1)}$ which will be helpful in the proof of Theorem 3.3.

Clearly, the natural action of $\mathrm{Aff}(n)$ on $\mathbb{R}^n$ is continuous. This action induces a continuous action on $2^{\mathbb{R}^n}$. Observe that for every $g \in \mathrm{Aff}(n)$ and $A \in cb(\mathbb{R}^n)$, the set $gA = \{ga \mid a \in A\}$ belongs to $cb(\mathbb{R}^n)$, i.e., $cb(\mathbb{R}^n)$ is an invariant subset of $2^{\mathbb{R}^n}$ and thus the restriction of the $\mathrm{Aff}(n)$-action on $cb(\mathbb{R}^n)$ is continuous. We will prove in Theorem 3.3 that this action is proper. First we prove the following two technical lemmas.

Lemma 3.1. Let $A \in cb(\mathbb{R}^n)$ and let $x_0 \in A$ be such that $\overline{N(x_0, 2\varepsilon)} \subset A$ for certain $\varepsilon > 0$. If $C \in O(A, \varepsilon)$ then $N(x_0, \varepsilon) \subset C$.

Proof. Suppose the contrary is true, i.e., that there exists $C \in O(A, \varepsilon)$ such that $N(x_0, \varepsilon) \not\subset C$. Choose $x \in N(x_0, \varepsilon) \setminus C$. Since C is compact, there exists $z \in C$ with $d(x, z) = d(x, C)$. Let H be the hyperplane through z in $\mathbb{R}^n$ orthogonal to the ray $\vec{xz}$. Since C is convex, it lies in the halfspace determined by H which does not contain the point x. Let a be the intersection point of the ray $\vec{zx}$ with the boundary $\partial\overline{N(x_0, 2\varepsilon)} \subset A$. Evidently, $d(a, x_0) = 2\varepsilon$ and

$$d(a, z) = d(a, H) \leq d(a, C) \leq d_H(A, C) < \varepsilon.$$

Since $d(x_0, x) < \varepsilon$ the triangle inequality implies that

$$\varepsilon > d(a, z) > d(a, x) \geq d(a, x_0) - d(x_0, x) > 2\varepsilon - \varepsilon = \varepsilon.$$

This contradiction proves the lemma. $\square$

Observe that $cb(\mathbb{R}^n)$ is not closed in $cc(\mathbb{R}^n)$. However, we have the following lemma:

Lemma 3.2. Let $A \in cb(\mathbb{R}^n)$ and $x_0 \in A$ be such that $\overline{N(x_0, 2\varepsilon)} \subset A$ for certain $\varepsilon > 0$. Then $\overline{O(A, \varepsilon)}$, the closure of $O(A, \varepsilon)$ in $cb(\mathbb{R}^n)$, is compact.

Proof. First we observe that $O(A, \varepsilon)$ is contained in $cc(K)$ for some compact convex subset $K \subset \mathbb{R}^n$, where $cc(K)$ stands for the hyperspace of all compact convex subsets of K. By [21], $cc(K)$ is compact, and hence, the closure of $O(A, \varepsilon)$ in $cc(K)$, denoted by $[O(A, \varepsilon)]$, is also compact. So, it is enough to prove that $[O(A, \varepsilon)]$ is contained in $cb(\mathbb{R}^n)$.

Let $(D_m)_{m\in\mathbb{N}} \subset O(A, \varepsilon)$ be a sequence of compact convex bodies converging to some $D \in cc(K)$. According to Lemma 3.1, $N(x_0, \varepsilon) \subset D_m$ for every $m \in \mathbb{N}$. Suppose that $N(x_0, \varepsilon) \not\subset D$. Pick an arbitrary point $x \in N(x_0, \varepsilon) \setminus D$ and let $\eta = d(x, D) > 0$. Since $x \in D_m$ for each $m \in \mathbb{N}$, it is clear that $d_H(D_m, D) \geq \eta$. This means that the sequence $(D_m)_{m\in\mathbb{N}}$ cannot converge to D, a contradiction. This contradiction proves that $N(x_0, \varepsilon)$ is contained in D, and therefore, D has a nonempty interior and then $D \in cb(\mathbb{R}^n)$.

Thus, $[O(A, \varepsilon)]$ is a compact set contained in $cb(\mathbb{R}^n)$ which yields that $\overline{O(A, \varepsilon)} = [O(A, \varepsilon)]$, and hence, $\overline{O(A, \varepsilon)}$ is compact.

□

Theorem 3.3. $\mathrm{Aff}(n)$ acts properly on $cb(\mathbb{R}^n)$.

Proof. Let $A \in cb(\mathbb{R}^n)$ and assume that $x_0 \in A$ and $\varepsilon > 0$ are such that $\overline{N(x_0, 2\varepsilon)} \subset A$. We claim that $O(A, \varepsilon)$ is a small neighborhood of A.

Indeed, let $B \in cb(\mathbb{R}^n)$. Since B has a nonempty interior, there is a point $z_0 \in B$ and $\delta > 0$ such that $\overline{N(z_0, 2\delta)} \subset B$. We will prove that the transporter

$$\Gamma = \{g \in \mathrm{Aff}(n) \mid gO(A, \varepsilon) \cap O(B, \delta) \neq \emptyset\}$$

has compact closure in $\mathrm{Aff}(n)$.

It is sufficient to prove that Γ, viewed as a subset of $\mathbb{R}^n \times \mathbb{R}^{n^2}$, is bounded and its closure in $\mathrm{Aff}(n)$ coincides with the one in $\mathbb{R}^n \times \mathbb{R}^{n^2}$.

For every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $\|x\|_\infty = \max\limits_{i=1}^{n} |x_i|$. There exists $M > 0$ such that, if $C \in O(A, \varepsilon) \cup O(B, \delta)$, then

$$\|c\|_\infty \leq M \quad \text{for all } c \in C. \tag{3.2}$$

In particular,

$$\operatorname{diam} C = \sup_{c, c' \in C} \|c - c'\|_\infty \leq 2M.$$

Take an arbitrary element $\mu \in \Gamma$. There exist $A' \in O(A, \varepsilon)$ and $B' \in O(B, \delta)$ with $\mu A' = B'$. Since μ is an affine transformation, there are $u \in \mathbb{R}^n$ and $\sigma \in GL(n)$ such that $\mu(x) = u + \sigma(x)$ for all $x \in \mathbb{R}^n$. Let (σ_{ij}) be the matrix representing σ with respect to the canonical basis of $\mathbb{R}^n$, and consider (σ_{ij}) as a point in $\mathbb{R}^{n^2}$.

Since $\mu A' = B' \in O(B, \delta)$, according to inequality (3.2), $\operatorname{diam} \mu A' \leq 2M$. Observe that $\mu A' = \sigma A' + u$, and hence, $\operatorname{diam} \sigma A' = \operatorname{diam} \mu A' \leq 2M$. Let

$$\xi_i = (0, \dots, 0, \varepsilon/2, 0, \dots, 0) \in \mathbb{R}^n,$$

where $\varepsilon/2$ is the *i-th* coordinate. Then, by Lemma 3.1, $\xi_i + x_0 \in N(x_0, \varepsilon) \subset A'$ and $-\xi_i + x_0 \in N(x_0, \varepsilon) \subset A'$. Since $\operatorname{diam} \sigma A' \leq 2M$, we get:

$$\begin{aligned} \|2\sigma(\xi_i)\|_\infty = \|\sigma(2\xi_i)\|_\infty &= \|\sigma((\xi_i + x_0) - (-\xi_i + x_0))\|_\infty \\ &= \|\sigma(\xi_i + x_0) - \sigma(-\xi_i + x_0)\|_\infty \leq 2M, \end{aligned}$$

and thus, $\|\sigma(\xi_i)\|_\infty \leq M$.

However, $\sigma(\xi_i) = (\sigma_{1i}\varepsilon/2, \dots, \sigma_{ni}\varepsilon/2)$, and therefore, $|\sigma_{ji}\varepsilon/2| \leq M$ for every $i = 1, \dots, n$ and $j = 1, \dots, n$. Thus, $|\sigma_{ji}| < 2M/\varepsilon$.

Next, according to (3.2), for every $a = (a_1, \dots, a_n) \in A'$ one has $\|a\|_\infty \leq M$. Then we get:

$$\|\sigma(a)\|_\infty = \max_{i=1}^{n} \Big| \sum_{j=1}^{n} \sigma_{ij} a_j \Big| \leq \sum_{i=1}^{n} \frac{2M}{\varepsilon} \|a\|_\infty \leq \frac{2nM^2}{\varepsilon}.$$

On the other hand, $\mu(a) \in B'$, which yields that

$$M \geq \|\mu(a)\|_\infty = \|u + \sigma(a)\|_\infty \geq \|u\|_\infty - \|\sigma(a)\|_\infty \geq \|u\|_\infty - \frac{2nM^2}{\varepsilon}.$$

This implies that $\|(u)\|_\infty \leq M + \frac{2nM^2}{\varepsilon}$, and therefore, Γ, viewed as a subset of $\mathbb{R}^n \times \mathbb{R}^{n^2}$, is bounded.

In order to complete the proof, it remains only to show that the closure of Γ in $\mathrm{Aff}(n)$ coincides with its closure in $\mathbb{R}^n \times \mathbb{R}^{n^2}$. Observe that here $\mathbb{R}^{n^2}$ represents the space of all real $n\times n$-matrices, i.e., $\mathbb{R}^{n^2}$ represents the space of all linear transformations from $\mathbb{R}^n$ into itself. Therefore, an element $\lambda \in \mathbb{R}^n \times \mathbb{R}^{n^2}$ represents a map which is the composition of a linear transformation followed by a translation. In this case, λ is an affine transformation iff it is surjective.

Let $(\lambda_m)_{m\in\mathbb{N}} \subset \Gamma$ be a sequence of affine transformations converging to some element $\lambda \in \mathbb{R}^n \times \mathbb{R}^{n^2}$. We need to prove that $\lambda \in \mathrm{Aff}(n)$. Since $\lambda_m \in \Gamma$, there exist $A_m \in O(A, \varepsilon)$ and $B_m \in O(B, \delta)$ such that $\lambda_m A_m = B_m$. According to Lemma 3.2, the closures $\overline{O(A, \varepsilon)}$ and $\overline{O(B, \delta)}$ are compact. Hence, we can assume that A_m converges to some $A_0 \in \overline{O(A, \varepsilon)}$ and B_m converges to some $B_0 \in \overline{O(B, \delta)}$. Then the equality $\lambda_m A_m = B_m$ yields that $\lambda A_0 = B_0$. Since B_0 has a nonempty interior, we infer that $\dim B_0 = n$, and hence, the dimension of the image $\lambda(\mathbb{R}^n)$ also equals n. Thus, $\lambda(\mathbb{R}^n)$ is an n-dimensional hyperplane in $\mathbb{R}^n$ which is possible only if $\lambda(\mathbb{R}^n) = \mathbb{R}^n$. Thus, λ is surjective, as required. This completes the proof. $\square$

3.2. **A convenient global slice for** $cb(\mathbb{R}^n)$**.** A well-known result of F. John [17] (see also [15]) in affine convex geometry states that for each $A \in cb(\mathbb{R}^n)$ there is a unique minimal-volume ellipsoid $l(A)$ containing A (respectively, a maximal-volume ellipsoid $j(A)$ contained in A). Nowadays $j(A)$ is called the *John ellipsoid* of A while $l(A)$ is called its *Löwner ellipsoid*. We will denote by $L(n)$ (resp., by $J(n)$) the subspace of $cb(\mathbb{R}^n)$ consisting of all convex bodies $A \in cb(\mathbb{R}^n)$ for which the Euclidean unit ball $\mathbb{B}^n$ is the Löwner ellipsoid (resp., the John ellipsoid). By $E(n)$ we will denote the subset of $cb(\mathbb{R}^n)$ consisting of all ellipsoids. Below we shall consider the map $l : cb(\mathbb{R}^n) \to E(n)$ that sends a convex body $A \in cb(\mathbb{R}^n)$ to its minimal-volume ellipsoid $l(A)$. We will call l the Löwner map.

Proposition 3.4. *$L(n)$ satisfies the following four properties:*

(a) *$L(n)$ is $O(n)$-invariant.*
(b) *The saturation $\mathrm{Aff}(n)\big(L(n)\big)$ coincides with $cb(\mathbb{R}^n)$.*
(c) *If $gL(n) \cap L(n) \neq \emptyset$ for some $g \in \mathrm{Aff}(n)$, then $g \in O(n)$.*
(d) *$L(n)$ is compact.*

Proof. First we prove the following
Claim. The Löwner map $l : cb(\mathbb{R}^n) \to E(n)$ is $\mathrm{Aff}(n)$-equivariant, i.e., $l(gA) = gl(A)$ for every $g \in \mathrm{Aff}(n)$ and $A \in cb(\mathbb{R}^n)$.

Assume the contrary is true, i.e., that there exist $A \in cb(\mathbb{R}^n)$ and $g \in \mathrm{Aff}(n)$ such that $l(gA) \neq gl(A)$. Clearly, $gl(A)$ is an ellipsoid containing gA. Since the minimal volume ellipsoid of $g(A)$ is unique, we infer that $\mathrm{vol}(gl(A)) > \mathrm{vol}(l(gA))$. By the same argument, $\mathrm{vol}(g^{-1}l(gA)) > \mathrm{vol}(l(A))$. Now we apply a well-known fact that each affine transformation preserves the ratio of volumes of any pair of compact convex bodies. Thus we obtain:

$$\frac{\mathrm{vol}(l(A))}{\mathrm{vol}(A)} = \frac{\mathrm{vol}(gl(A))}{\mathrm{vol}(gA)} > \frac{\mathrm{vol}(l(gA))}{\mathrm{vol}(gA)} = \frac{\mathrm{vol}(g^{-1}l(gA))}{\mathrm{vol}(A)} > \frac{\mathrm{vol}(l(A))}{\mathrm{vol}(A)}.$$

This contradiction proves the claim.

(a) Let $g \in O(n)$ and $A \in L(n)$. The above claim implies that $l(gA) = gl(A) = g\mathbb{B}^n = \mathbb{B}^n$, i.e., $gA \in L(n)$, which means that $L(n)$ is $O(n)$-invariant.

(b) Let $A \in cb(\mathbb{R}^n)$. There exists $g \in \mathrm{Aff}(n)$ such that $l(A) = g\mathbb{B}^n$. According to the above claim we have:

$$\mathbb{B}^n = g^{-1}l(A) = l(g^{-1}A).$$

Then, $g^{-1}A \in L(n)$ and $A = g(g^{-1}A)$. This proves that $\mathrm{Aff}(n)\big(L(n)\big) = cb(\mathbb{R}^n)$.

(c) If there exist $g \in \mathrm{Aff}(n)$ and $A \in L(n)$ such that $gA \in L(n)$, then

$$\mathbb{B}^n = l(gA) = gl(A) = g\mathbb{B}^n.$$

This yields that $g \in O(n)$.

(d) Clearly, $L(n) \subset cc(\mathbb{B}^n)$. Since $cc(\mathbb{B}^n)$ is compact (in fact, it is homeomorphic to the Hilbert cube [21, Theorem 2.2]), it suffices to show that $L(n)$ is closed in $cc(\mathbb{B}^n)$.

Let $(A_k)_{k\in\mathbb{N}} \subset L(n)$ be a sequence converging to $A \in cc(\mathbb{B}^n)$. We will prove that $A \in L(n)$. To this end, we shall prove first that A has nonempty interior. If not, there exist an $(n-1)$-dimensional hyperplane $\mathcal{H} \subset \mathbb{R}^n$ such that $A \subset \mathcal{H}$. Let $E' \subset \mathcal{H}$ be an $(n-1)$-dimensional ellipsoid containing A in its interior (with respect to $\mathcal{H}$). For any $r > 0$, consider the line segment

T_r of length r which is orthogonal to $\mathcal{H}$ and passes trough the center of E'. Let $r > 0$ be small enough that the n-dimensional ellipsoid E generated by E' and T_r has the volume less than $\operatorname{vol}(\mathbb{B}^n)$. Since A lies in the interior of E, there exist $\delta > 0$ such that $N(A, \delta) \subset E$. Now, we use the fact that (A_k) converges to A to find $m_0 \in \mathbb{N}$ such that $A_{m_0} \subset N(A, \delta) \subset E$. Thus, E is an ellipsoid containing A_{m_0} and then

$$\operatorname{vol}(\mathbb{B}^n) = \operatorname{vol}\,(l(A_{m_0})) < \operatorname{vol}(E) < \operatorname{vol}(\mathbb{B}^n).$$

This contradiction proves that A has nonempty interior.

Consequently, $l(A)$ is defined and we have to show that $l(A) = \mathbb{B}^n$. Suppose that $l(A) \neq \mathbb{B}^n$. Since $A_k \subset \mathbb{B}^n$ for every $k \in \mathbb{N}$, it follows that $A \subset \mathbb{B}^n$. Hence, by uniqueness of the minimal volume ellipsoid, $\operatorname{vol}(l(A)) < \operatorname{vol}(\mathbb{B}^n)$. Let L be an ellipsoid concentric and homothetic with $l(A)$ with ratio > 1 and $\operatorname{vol}(L) < \operatorname{vol}(\mathbb{B}^n)$. As $l(A)$ is contained in the interior of L, the distance $d_H(\partial L, \partial l(A)) = \varepsilon$ is positive. Consider $U = N\big(\partial l(A), \varepsilon\big)$, the ε-neighborhood of the boundary $\partial l(A)$ in $\mathbb{R}^n$. Since $(A_k)_{k\in\mathbb{N}}$ converges to A and all the sets A_k are convex, the sequence $(\partial A_k)_{k\in\mathbb{N}}$ converges to ∂A. Therefore, there exists $k_0 \geq 1$ such that $\partial A_{k_0} \subset U$. The convexity of A_{k_0} implies that $A_{k_0} \subset L$, and hence,

$$\operatorname{vol}\big(l(A_{k_0})\big) \leq \operatorname{vol}(L) < \operatorname{vol}(\mathbb{B}^n) = \operatorname{vol}\big(l(A_{k_0})\big).$$

This contradiction proves that $A \in L(n)$, and hence, $L(n)$ is closed in $cc(\mathbb{B}^n)$. □

Remark 3.5. The first three assertions of Proposition 3.4 are easy modifications of those in [6, Proof of Theorem 4], while the forth one provides a new way of proving Macbeath's result on compactness of the orbit space $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ (see Corollary 3.7(1)).

Theorem 3.6. (1) The Löwner map $l : cb(\mathbb{R}^n) \to E(n)$ is an $\operatorname{Aff}(n)$-equivariant retraction with $L(n) = l^{-1}(\mathbb{B}^n)$.

(2) $L(n)$ is a compact global $O(n)$-slice for the proper $\operatorname{Aff}(n)$-space $cb(\mathbb{R}^n)$.

Proof. (1) In the proof of Proposition 3.4 we showed that $l : cb(\mathbb{R}^n) \to E(n)$ is $\operatorname{Aff}(n)$-equivariant. Clearly, it is a retraction. As to the continuity of l, it is a standard consequence of the above four properties in Proposition 3.4, well known in transformation groups (see [12, Ch. II, Theorem 4.2 and Theorem 4.4] for compact group actions and [23] for locally compact proper group actions). However, using the compactness of $L(n)$ we shall give here a direct proof of this fact.

For, let $(X_m)_{m=1}^{\infty}$ be a sequence in $cb(\mathbb{R}^n)$ that converges to a point $X \in cb(\mathbb{R}^n)$, i.e., $X_m \rightsquigarrow X$. We must show that $l(X_m) \rightsquigarrow l(X)$. Assume the contrary is true. Then there exist a number $\varepsilon > 0$ and a subsequence (A_k) of the sequence (X_m) such that $d_H\big(l(A_k), l(A)\big) \geq \varepsilon$ for all $k = 1, 2, \dots,$ where d_H denotes the Hausdorff metric.

By property (b) of Proposition 3.4, there are $g, g_k \in \operatorname{Aff}(n)$, $k = 1, 2, \dots,$ such that $A_k = g_k S_k$ and $A = gP$ for some $P, S_k \in L(n)$. Due to compactness of $L(n)$, without loss of generality, one can assume that $S_k \rightsquigarrow S$ for some $S \in L(n)$. Since $\operatorname{Aff}(n)$ acts properly on $cb(\mathbb{R}^n)$ (see Theorem 3.3), the points S and P have neighborhoods U_S and U_P, respectively, such that the transporter $\langle U_S, U_P \rangle$ has compact closure. Since $S_k \rightsquigarrow S$ and $g^{-1}g_k S_k \rightsquigarrow P$, it then follows that there is a natural number k_0 such that $g^{-1}g_k \in \langle U_S, U_P \rangle$ for all $k \geq k_0$. Consequently, the sequence $(g^{-1}g_k)$ has a convergent subsequence. Again, it is no loss of generality to assume that $g^{-1}g_k \rightsquigarrow h$ for some $h \in \operatorname{Aff}(n)$. This implies that $g^{-1}g_k S_k \rightsquigarrow hS$, which together with $g^{-1}g_k S_k \rightsquigarrow P$ yields that $hS = P$. But S and P belong to $L(n)$, and hence, property (c) of Proposition 3.4 yields that $h \in O(n)$. Since $g_k \rightsquigarrow gh$, then we get that

$$l(A_k) = l(g_k S_k) = g_k l(S_k) = g_k \mathbb{B}^n \rightsquigarrow gh\mathbb{B}^n = g\mathbb{B}^n = gl(S) = l(gS) = l(A),$$

which contradicts to the inequality $d_H\big(l(A_k), l(A)\big) \geq \varepsilon$, $k = 1, 2, \dots.$

Hence, $l(X_m) \rightsquigarrow l(X)$, as required.

(2) Compactness of $L(n)$ was proved in Proposition 3.4(d). Since $E(n)$ is the $\operatorname{Aff}(n)$-orbit of the point $\mathbb{B}^n \in cb(\mathbb{R}^n)$ and $O(n)$ is the stabilizer of $\mathbb{B}^n$, one has the $\operatorname{Aff}(n)$-homeomorphism $E(n) \cong \operatorname{Aff}(n)/O(n)$ (see [23, Proposition 1.1.5]). This, together with the statement (1), yields an $\operatorname{Aff}(n)$-equivariant map $f\colon cb(\mathbb{R}^n) \to \operatorname{Aff}(n)/O(n)$ such that $L(n) = f^{-1}\big(O(n)\big)$. Thus, $L(n)$ is a global $O(n)$-slice for $cb(\mathbb{R}^n)$, as required.

□

Corollary 3.7. (1) (Macbeath [20]) The $\operatorname{Aff}(n)$-orbit space $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ is compact.

(2) The two orbit spaces $L(n)/O(n)$ and $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ are homeomorphic.

Proof. Let $\pi\colon L(n) \to cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ be the restriction of the orbit map $cb(\mathbb{R}^n) \to cb(\mathbb{R}^n)/\operatorname{Aff}(n)$. Then π is continuous and it follows from Proposition 3.4(b) that π is onto. This already implies the first assertion if we remember that $L(n)$ is compact (see Proposition 3.4(d)).

Further, for $A, B \in L(n)$, it follows from Proposition 3.4(c) that $\pi(A) = \pi(B)$ iff A and B have the same $O(n)$-orbit. Hence, π induces a continuous bijective map $p\colon L(n)/O(n) \to cb(\mathbb{R}^n)/\operatorname{Aff}(n)$. Since $L(n)/O(n)$ is compact we then conclude that p is a homeomorphism.

□

In Theorem 5.11 we will prove that the orbit space $L(n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$. This, in combination with Corollary 3.7 implies the following:

Corollary 3.8. The $\operatorname{Aff}(n)$-orbit space $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ is homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$.

Corollary 3.9. (1) There exists an $O(n)$-equivariant retraction $r\colon cb(\mathbb{R}^n) \to L(n)$ such that $r(A)$ belongs to the $\operatorname{Aff}(n)$-orbit of A.

(2) The diagonal product of the two retractions $r\colon cb(\mathbb{R}^n) \to L(n)$ and $l\colon cb(\mathbb{R}^n) \to E(n)$ is an $O(n)$-equivariant homeomorphism $cb(\mathbb{R}^n) \cong_{O(n)} L(n) \times E(n)$.

Proof. (1) Recall that $O(n)$ is a maximal compact subgroup of $\operatorname{Aff}(n)$. According to the structure theorem (see [16, Ch. XV, Theorem 3.1]), there exists a closed subset $T \subset \operatorname{Aff}(n)$ such that $gTg^{-1} = T$ for every $g \in O(n)$, and the multiplication map

$$(t, g) \mapsto tg\colon T \times O(n) \to \operatorname{Aff}(n) \tag{3.3}$$

is a homeomorphism. In our case it is easy to see that T can be taken as the set of all products AS, where A is a translation and S is a non-degenerate symmetric (or self-adjoint) positive operator. This follows easily from two standard facts in Linear Algebra: (1) each $a \in \operatorname{Aff}(n)$ is uniquely represented as the composition of a translation $t \in \mathbb{R}^n$ and an invertible operator $g \in GL(n)$, (2) due to the polar decomposition theorem, every invertible operator $g \in GL(n)$ can uniquely be represented as the composition of a non-degenerate symmetric positive operator and an orthogonal operator (see, e.g., [18, sections 2.3 and 2.4]).

Now we define the required $O(n)$-equivariant retraction $r\colon cb(\mathbb{R}^n) \to L(n)$.

Let $f\colon \operatorname{Aff}(n) \to E(n)$ be defined by $f(g) = g\mathbb{B}^n$. Then f induces an $\operatorname{Aff}(n)$-equivariant homeomorphism $\widetilde{f}\colon \operatorname{Aff}(n)/O(n) \to E(n)$ [23, Proposition 1.1.5] and it is the composition of the following two maps:

$$\operatorname{Aff}(n) \xrightarrow{\pi} \operatorname{Aff}(n)/O(n) \xrightarrow{\widetilde{f}} E(n)$$

where π is the natural quotient map. Due to compactness of $O(n)$, π is closed, and hence, f being the composition of two closed maps is itself closed.

This yields that the restriction $f|_T\colon T \to E(n)$ is a homeomorphism. Moreover, this homeomorphism is $O(n)$-equivariant if we let $O(n)$ act on T by inner automorphisms and on $E(n)$ by the action induced from $cb(\mathbb{R}^n)$.

Denote by $\xi\colon E(n) \to T$ the inverse map f^{-1}. Then we have the following characteristic property of ξ:

$$[\xi(C)]^{-1}C = \mathbb{B}^n \quad \text{for all} \quad C \in E(n). \tag{3.4}$$

Next, we define

$$r(A) = [\xi(l(A))]^{-1}A \quad \text{for every} \quad A \in cb(\mathbb{R}^n).$$

Clearly, r depends continuously on $A \in cb(\mathbb{R}^n)$.

Since $l\big(r(A)\big) = l\big([\xi(l(A))]^{-1}A\big) = [\xi(l(A))]^{-1}l(A)$ and, since by (3.4), $[\xi(l(A))]^{-1}l(A) = \mathbb{B}^n$, we infer that $r(A) \in L(n)$. If $A \in L(n)$, then $l(A) = \mathbb{B}^n$ and $r(A) = [\xi(l(A))]^{-1}A = [\xi(\mathbb{B}^n)]^{-1}A = 1 \cdot A = A$. Thus, r is a well-defined retraction on $L(n)$.

Let us check that it is $O(n)$-equivariant. For, let $g \in O(n)$ and $A \in cb(\mathbb{R}^n)$. Then $r(gA) = [\xi(l(gA))]^{-1}gA = [\xi(gl(A))]^{-1}gA$. Due to equivariance of ξ, one has $\xi(gl(A)) = g\xi(l(A))g^{-1}$, and hence, $[\xi(gl(A))]^{-1} = g[\xi(l(A))]^{-1}g^{-1}$. Consequently,

$$r(gA) = \Big(g[\xi(l(A))]^{-1}g^{-1}\Big)gA = g\Big([\xi(l(A))]^{-1}A\Big) = gr(A),$$

as required. Thus, $r\colon cb(\mathbb{R}^n) \to L(n)$ is an $O(n)$-retraction, and clearly, $r(A)$ belongs to the $\mathrm{Aff}(n)$-orbit of A.

(2) Next we define

$$\varphi(A) = \big(r(A),\ l(A)\big) \quad \text{for every} \quad A \in cb(\mathbb{R}^n).$$

Then φ is the desired $O(n)$-equivariant homeomorphism $cb(\mathbb{R}^n) \to L(n) \times E(n)$ with the inverse map given by $\varphi^{-1}\big((C,E)\big) = \xi(E)C$ for every pair $(C,E) \in L(n) \times E(n)$.

$\square$

Corollary 3.10. (1) $E(n)$ is an $O(n)$-AR.

(2) $E(n)$ is homeomorphic to the Euclidean space $\mathbb{R}^{n(n+3)/2}$.

Proof. (1) Follows immediately from Theorem 3.6 and from the fact that $cb(\mathbb{R}^n)$ is an $O(n)$-AR [8, Corollary 4.8].

(2) As we observed above, $E(n)$ is homeomorphic to the quotient space $\mathrm{Aff}(n)/O(n)$ (see [23, Proposition 1.1.5]). Consequently, one should prove that $\mathrm{Aff}(n)/O(n)$ is homeomorphic to $\mathbb{R}^{n(n+3)/2}$.

Since $\mathrm{Aff}(n)$ is the semidirect product of $\mathbb{R}^n$ and $GL(n)$, as a topological space $\mathrm{Aff}(n)/O(n)$ is homeomorphic to $\mathbb{R}^n \times GL(n)/O(n)$. The *RQ*-decomposition theorem in Linear Algebra states that every invertible matrix can uniquely be represented as the product of an orthogonal matrix and an upper-triangular matrix with positive elements on the diagonal (see, e.g., [13, Fact 4.2.2 and Exercise 4.3.29]). This easily yields that $GL(n)/O(n)$ is homeomorphic to $\mathbb{R}^{(n+1)n/2}$, and hence, $\mathrm{Aff}(n)/O(n)$ is homeomorphic to $\mathbb{R}^p$, where $p = n + (n+1)n/2 = n(n+3)/2$.

$\square$

In Section 5 we will prove that $L(n)$ is homeomorphic to the Hilbert cube (see Corollary 5.9). This, in combination with Corollaries 3.9 and 3.10, yields the following result, which is one of the main results of the paper:

Corollary 3.11. $cb(\mathbb{R}^n)$ is homeomorphic to $Q \times \mathbb{R}^{n(n+3)/2}$.

Remark 3.12. Using the maximal-volume ellipsoids instead of the minimal-volume ellipsoids, one can prove in a similar way that the subset $J(n)$, defined at the beginning of this subsection, is also a global $O(n)$-slice for $cb(\mathbb{R}^n)$. However, it follows from a result of H. Abels [1, Lemma 2.3] that the two global $O(n)$-slices $J(n)$ and $L(n)$ are equivalent in the sense that there exists an $\mathrm{Aff}(n)$-equivariant homeomorphism $f\colon cb(\mathbb{R}^n) \to cb(\mathbb{R}^n)$ such that $f\big(L(n)\big) = J(n)$. Consequently, all the results stated in terms of $L(n)$ have also their dual analogs in terms of $J(n)$, which can be proven by trivial modification of our proofs of the corresponding "$L(n)$-results".

4. The hyperspace $M(n)$

Let us denote by $M(n)$ the $O(n)$-invariant subspace of $cc(\mathbb{R}^n)$ consisting of all $A \in cc(\mathbb{R}^n)$ such that $\max\limits_{a\in A} \|a\| = 1$. Thus, $M(n)$ consists of all compact convex subsets of $\mathbb{B}^n$ which intersect the boundary sphere $\mathbb{S}^{n-1}$.

It is evident that $M(n)$ is closed in $cc(\mathbb{B}^n) \subset cc(\mathbb{R}^n)$. Due to compactness of $cc(\mathbb{B}^n)$ (a well-known fact) it then follows that $M(n)$ is compact as well. The importance of $M(n)$ lies in the property that $cc(\mathbb{R}^n)$ is the open cone over it (see Section 7). In this section we will prove that $M(n)$ is also homeomorphic to the Hilbert cube (Corollary 4.13) and its orbit space $M(n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$ (Theorem 4.16).

Let us recall that a G-space X is called *strictly G-contractible* if there exists a G-homotopy $F : X \times [0,1] \to X$ and a G-fixed point $a \in X$ such that $F(x,0) = x$ for all $x \in X$ and $F(x,t) = a$ if and only if $t = 1$ or $x = a$.

Lemma 4.1. $M(n)$ is strictly $O(n)$-contractible to its only $O(n)$-fixed point $\mathbb{B}^n$.

Proof. The map $F : M(n) \times [0,1] \to M(n)$ defined by

$$F(A,t) = (1-t)A + t\mathbb{B}^n$$

is the desired $O(n)$-contraction. □

Consider the map $\nu : cc(\mathbb{R}^n) \to [0,\infty)$ defined by

$$\nu(A) = \max_{a \in A} \|a\|, \quad A \in cc(\mathbb{R}^n). \tag{4.1}$$

Lemma 4.2. ν is a uniformly continuous $O(n)$-invariant map.

Proof. Let $\varepsilon > 0$, $A, B \in cc(\mathbb{R}^n)$ and suppose that $d_H(A,B) < \varepsilon$. Let $a \in A$ be such that $\nu(A) = \|a\|$. Then there exists a point $b \in B$ with $\|a - b\| < \varepsilon$. Since $\|b\| \leq \nu(B)$ we have the following inequalities:

$$\varepsilon > \|a - b\| \geq \|a\| - \|b\| \geq \nu(A) - \nu(B).$$

Similarly, we can prove that $\nu(B) - \nu(A) < \varepsilon$, and hence, ν is uniformly continuous.

Now, if $g \in O(n)$ then $\|gx\| = \|x\|$ for every $x \in \mathbb{R}^n$. Thus,

$$\nu(gA) = \max_{a' \in gA} \|a'\| = \max_{a \in A} \|ga\| = \max_{a \in A} \|a\| = \nu(A).$$

This proves that ν is $O(n)$-invariant, as required.

□

Lemma 4.3. $M(n)$ is an $O(n)$-AR with a unique $O(n)$-fixed point, $\mathbb{B}^n$.

Proof. By [8, Corollary 4.8], $cc(\mathbb{R}^n)$ is an $O(n)$-AR. Hence, the complement $cc(\mathbb{R}^n) \setminus \{0\}$ is an $O(n)$-ANR. The map $r : cc(\mathbb{R}^n) \setminus \{0\} \to M(n)$ defined by the rule:

$$r(A) = \frac{1}{\nu(A)} A \tag{4.2}$$

is an $O(n)$-retraction, where ν is the map defined in (4.1). Thus $M(n)$, being an $O(n)$-retract of an $O(n)$-ANR, is itself an $O(n)$-ANR. On the other hand, it was shown in Lemma 4.1 that $M(n)$ is $O(n)$-contractible to its point $\mathbb{B}^n$. Since every $O(n)$-contractible $O(n)$-ANR space is $O(n)$-AR (see [3]) we conclude that $M(n)$ is an $O(n)$-AR. This completes the proof. □

The following lemma will be used several times throughout the rest of the paper:

Lemma 4.4. Let $p_1, \dots, p_k \in \mathbb{R}^n$ be a finite number of points. Let $K \subset O(n)$ be a closed subgroup which acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$. Then the boundary ∂D of the convex hull

$$D = \operatorname{conv}\big(K(p_1) \cup \cdots \cup K(p_k)\big)$$

does not contain an $(n-1)$-dimensional elliptic domain, i.e., ∂D does not contain an open subset $V \subset \partial D$ which at the same time is an open connected subset of some $(n-1)$-dimensional ellipsoid surface lying in $\mathbb{R}^n$.

Proof. Assume the contrary, that there exists an open subset $V \subset \partial D$ of the boundary ∂D which is an $(n-1)$-dimensional elliptic domain. Recall that a convex body $A \subset \mathbb{R}^n$ is called strictly convex, if every boundary point $a \in \partial A$ is an extreme point; that is to say that the complement $A \setminus \{a\}$ is convex. Since every ellipsoid in $\mathbb{R}^n$ is strictly convex, we conclude that every point $v \in V$ is an extreme point for D too. This is easy to show.

Indeed, suppose that there are two distinct points $b, c \in D$ such that v belongs to the relative interior of the line segment $[b, c] = \{\lambda b + (1-\lambda)c \mid \lambda \in [0, 1]\}$. Since v is a boundary point of D, it then follows that the whole segment $[b, c]$ lies in the boundary ∂D. Next, since V is open in ∂D, we infer that for b and c sufficiently close to v, the line segment $[b, c]$ is contained in V. However, this is impossible because V is an elliptic domain.

Thus, we have proved that every point $v \in V$ is an extreme point for D. Next, since D is the convex hull of the set $\bigcup\limits_{1=1}^{k} K(p_i)$, each extreme point of D lies in $\bigcup\limits_{1=1}^{k} K(p_i)$ (see, e.g., [29, Corollary 2.6.4]). This implies that V is contained in the union $\bigcup\limits_{1=1}^{k} K(p_i)$. Further, due to connectedness of V, it then follows that V is contained in only one $K(p_i)$. Next, let us show that this is impossible.

Indeed, since $K(p_i)$ lies on the $(n-1)$-sphere $\partial N(0, \|p_i\|)$ centered at the origin and having the radius $\|p_i\|$, the set V should be a domain of this sphere. As $K(p_i)$ is a homogeneous compact space, there exists a finite cover $\{V_1, \dots, V_m\}$ of $K(p_i)$, where each V_j is homeomorphic to V. Then, by the Domain Invariance Theorem (see, e.g., [25, Ch. 4, Section 7, Theorem 16]), each V_j is open in the sphere $\partial N(0, \|p_i\|)$. Hence, the union $V_1 \cup \cdots \cup V_m = K(p_i)$ is open in the sphere $\partial N(0, \|p_i\|)$. But $K(p_i)$ is also compact, and therefore, closed in $\partial N(0, \|p_i\|)$. Thus $K(p_i)$ is an open and closed subset of the connected space $\partial N(0, \|p_i\|)$, and consequently, $K(p_i) = \partial N(0, \|p_i\|)$. This yields that K acts transitively on the unit sphere $\mathbb{S}^{n-1}$, which is a contradiction. $\square$

The Fell topology in $cc(\mathbb{R}^n)$ is the topology generated by the sets of the form:

$$U^- = \{A \in cc(\mathbb{R}^n) \mid A \cap U \neq \emptyset\} \quad \text{and}$$
$$(\mathbb{R}^n \setminus K)^+ = \{A \in cc(\mathbb{R}^n) \mid A \subset \mathbb{R}^n \setminus K\},$$

where $U \subset \mathbb{R}^n$ is open and $K \subset \mathbb{R}^n$ is compact.

It is well known that the Fell topology and the Hausdorff metric topology coincide in $cc(\mathbb{R}^n)$ (see, e.g., [24, Remark 2]). In particular, both topologies coincide in $cb(\mathbb{R}^n)$. This fact will be used in the proof of the following lemma:

Lemma 4.5. Let $T \in cb(\mathbb{R}^n)$ be a convex body and $\mathcal{H} \subset cb(\mathbb{R}^n)$ a subset such that for every $A \in \mathcal{H}$, the intersection $A \cap T$ has nonempty interior. Then the map $\upsilon : \mathcal{H} \to cb(\mathbb{R}^n)$ defined by

$$\upsilon(A) = A \cap T, \qquad A \in \mathcal{H}$$

is continuous.

Proof. It is enough to show that $\upsilon^{-1}(U^-)$ and $\upsilon^{-1}\big((\mathbb{R}^n \setminus K)^+\big)$ are open in $\mathcal{H}$ for every open $U \subset \mathbb{R}^n$ and compact $K \subset \mathbb{R}^n$.

First, suppose that $U \subset \mathbb{R}^n$ is open and $A \in \upsilon^{-1}(U^-)$. Then $U \cap (A \cap T) \neq \emptyset$. Since U is open and $A \cap T$ is a convex body, there exists a point x_0 in the interior of $A \cap T$ such that $x_0 \in U$. So, one can find $\delta > 0$ satisfying

$$\overline{N(x_0, 2\delta)} \subset U \cap (A \cap T).$$

In accordance with Lemma 3.1, if $C \in O(A, \delta) \cap \mathcal{H}$ then $N(x_0, \delta) \subset C$. Since $x_0 \in U \cap T$, we conclude that $U \cap \upsilon(C) = U \cap (C \cap T) \neq \emptyset$. This proves that $O(A, \delta) \cap \mathcal{H} \subset \upsilon^{-1}(U^-)$, and hence, $\upsilon^{-1}(U^-)$ is open in $\mathcal{H}$.

Consider now a compact subset $K \subset \mathbb{R}^n$ and suppose $A \in \mathcal{H}$ is such that $\upsilon(A) \cap K = \emptyset$. If $K \cap T = \emptyset$ then $\mathcal{H} = \upsilon^{-1}\big((\mathbb{R}^n \setminus K)^+\big)$ which is open in $\mathcal{H}$. If $K \cap T \neq \emptyset$ then we define

$$\eta = \inf\,\{d(a, x) \mid a \in A,\ x \in K \cap T\}.$$

Since $(A \cap T) \cap K = \emptyset$, we have that $\eta > 0$. Let $C \in O(A, \eta) \cap \mathcal{H}$ and suppose that $\upsilon(C)$ meets K. Then there exists a point $x_0 \in C \cap T \cap K$. Since C belongs to the η-neighborhood of A, we can find a point $a \in A$ such that $d(a, x_0) < \eta$, contradicting to the choice of η. Then we conclude that

$$O(A, \eta) \cap \mathcal{H} \subset \upsilon^{-1}\big((\mathbb{R}^n \setminus K)^+\big),$$

and hence, $\upsilon^{-1}\big((\mathbb{R}^n \setminus K)^+\big)$ is open in $\mathcal{H}$. This completes the proof. $\square$

Denote by $M_0(n)$ the complement $M(n) \setminus \{\mathbb{B}^n\}$.

Proposition 4.6. For each closed subgroup $K \subset O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$ and each $\varepsilon > 0$, there exists a K-equivariant map $\chi_\varepsilon : M(n) \to M_0(n)$ which is ε-close to the identity map of $M(n)$. In particular, $\chi_\varepsilon\big(M(n)^K\big) \subset M_0(n)^K$.

Proof. Let $r : cc(\mathbb{R}^n) \setminus \{0\} \to M(n)$ be the $O(n)$-equivariant retraction defined in (4.2). Since $M(n)$ is compact, one can find a real $0 < \delta < \varepsilon/2$ such that $d_H\big(r(A), A\big) < \varepsilon/2$ for all A belonging to the δ-neighborhood of $M(n)$ in $cc(\mathbb{R}^n) \setminus \{0\}$.

Chose a convex polyhedron $P \subset \mathbb{B}^n$ with nonempty interior, $\delta/4$-close to $\mathbb{B}^n$ such that all the vertices $p_1, \dots, p_k$ of P lie on the unit sphere $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. Then the convex hull

$$T = \operatorname{conv}\big(K(p_1) \cup \dots \cup K(p_k)\big)$$

is a compact convex K-invariant subset of $\mathbb{R}^n$. By Lemma 4.4, the boundary ∂T does not contain an $(n-1)$-dimensional elliptic domain. Furthermore,

$$d_H(\mathbb{B}^n, T) \le d_H(\mathbb{B}^n, P) < \delta/4. \tag{4.3}$$

Let $h : M(n) \to M(n)$ be defined as follows:

$$h(A) = \{x \in \mathbb{B}^n \mid d(x, A) \le \delta/2\}, \quad \text{for every } A \in M(n).$$

Clearly, $h(A) \cap T$ is a nonempty set with a nonempty interior.

Then setting

$$\chi'(A) = h(A) \cap T$$

we obtain a map $\chi' : M(n) \to cc(\mathbb{R}^n)$. Since T is a K-fixed point of $cc(\mathbb{R}^n)$, we see that χ' is K-equivariant.

Continuity of χ' follows from the one of h and Lemma 4.5.

We claim that for any $A \in M(n)$, $\chi'(A)$ is not a closed Euclidean ball centered at the origin.

Indeed, if $h(A) \subset T$ then $h(A) \neq \mathbb{B}^n$ since T is strictly contained in $\mathbb{B}^n$. In this case $\chi'(A) = h(A) \cap T = h(A)$, and hence, $\chi'(A) \in M(n)$. However, the only Euclidean ball centered at the origin that belongs to $M(n)$ is $\mathbb{B}^n$. But $\chi'(A) = h(A) \neq \mathbb{B}^n$.

If $h(A)$ is not contained in T, then the boundary of $\chi'(A)$ contains a domain lying in the boundary ∂T. Since the boundary ∂T does not contain an $(n-1)$-dimensional elliptic domain (as shown in Lemma 4.4), we conclude that $\chi'(A)$ is not an ellipsoid. In particular, $\chi'(A)$ is not a Euclidean ball centered at the origin, and the claim is proved.

Now we assert that the composition $\chi = r \circ \chi'$ is the desired map. Indeed, $r(A) = \mathbb{B}^n$ if and only if A is a Euclidean ball centered at the origin. Since $\chi'(A)$ is not a Euclidean ball centered at the origin, we infer that $\chi(A) = r(\chi'(A)) \neq \mathbb{B}^n$ for every $A \in M(n)$. Thus $\chi : M(n) \to M_0(n)$ is a well-defined map. It is continuous and K-equivariant because χ' and r are so.

Now, if $x \in \chi'(A)$ then $x \in h(A)$. Hence, $d(x, A) \leq \delta/2 < \delta$ and $\chi'(A) \subset N(A, \delta)$. On the other hand, if $a \in A \subset \mathbb{B}^n$, then due to (4.3) there exists a point $x \in T$ such that $d(x, a) < \delta/4 < \delta/2$. Therefore, $x \in h(A) \cap T = \chi'(A)$, and hence, $A \subset N\big(\chi'(A), \delta/2\big)$. This proves that $d_H\big(A, \chi'(A)\big) < \delta$.

By the choice of δ the last inequality implies that $d_H\big(r(\chi'(A)), \chi'(A)\big) \leq \varepsilon/2$. Then for all $A \in M(n)$ we have:

$$\begin{aligned} d_H\big(\chi(A), A\big) &\leq d_H\big(\chi(A), \chi'(A)\big) + d_H\big(\chi'(A), A\big) \\ &= d_H\big(r(\chi'(A)), \chi'(A)\big) + d_H\big(\chi'(A), A\big) \\ &< \varepsilon/2 + \delta < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves that χ is ε-close to the identity map of $M(n)$, and the proof is now complete.

□

Observe that the induced action of $O(n)$ on $cc(\mathbb{R}^n)$ is isometric with respect to the Hausdorff metric. In particular, for every closed subgroup $K \subset O(n)$, the Hausdorff metric on $cc(\mathbb{R}^n)$ is K-invariant.

Let d_H^* be the metric on $M(n)/K$ induced by the Hausdorff metric on $M(n)$ as defined in equation (2.1):

$$d_H^*\big(K(A), K(B)\big) = \inf_{k \in K} d_H(A, kB), \quad A, B \in M(n).$$

Corollary 4.7. Let $K \subset O(n)$ be a closed subgroup that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$ then

(1) the singleton $\{\mathbb{B}^n\}$ is a Z-set in $M(n)^K$,
(2) the class of $\{\mathbb{B}^n\}$ is a Z-set in $M(n)/K$.

Proof. The first statement follows directly from Proposition 4.6. For the second statement take $\varepsilon > 0$. By Proposition 4.6, there exists a K-map $\chi_\varepsilon : M(n) \to M_0(n)$ such that $d_H(A, \chi(A)) < \varepsilon$ for every $A \in M(n)$. This induces a continuous map $\widetilde{\chi}_\varepsilon : M(n)/K \to M_0/K$ as follows:

$$\widetilde{\chi}_\varepsilon\big(K(A)\big) = \pi\big(\chi_\varepsilon(A)\big) = K\big(\xi_\varepsilon(A)\big), \quad A \in M(n),$$

where $\pi : M(n) \to M(n)/K$ is the K-orbit map. According to inequality (2.2) we have:

$$d_H^*\big(K(\chi_\varepsilon(A)), K(A)\big) \leq d_H\big(\chi_\varepsilon(A), A\big) < \varepsilon$$

and thus, $\widetilde{\chi}_\varepsilon$ is ε-close to the identity map of $M(n)/K$.

On the other hand, since $\{\chi_\varepsilon(A)\} \neq \{\mathbb{B}^n\} = K(\mathbb{B}^n)$ for every $A \in M(n)$, we conclude that

$$\widetilde{\chi}_\varepsilon\big(M(n)/K\big) \cap \{\mathbb{B}^n\} = \emptyset,$$

which proves that the class of $\{\mathbb{B}^n\}$ is a Z-set on $M(n)/K$. $\square$

Now, we shall give a sequence of lemmas and propositions culminating in Corollary 4.15

Denote by $\mathcal{R}(n)$ the subspace of $M(n)$ consisting of all $A \in M(n)$ such that the contact set $A \cap \mathbb{S}^{n-1}$ has empty interior in $\mathbb{S}^{n-1}$.

For every $A \in M(n)$, the intersection $A \cap \mathbb{S}^{n-1}$ is nonempty, and therefore, there exists a point $a \in A \cap \mathbb{S}^{n-1}$. If $O(n)_A$ is the $O(n)$-stabilizer of A then $O(n)_A(a) \subset A \cap \mathbb{S}^{n-1}$. Therefore, if A is different from $\mathbb{B}^n$, the subset $O(n)_A(a)$ should be different from $\mathbb{S}^{n-1}$, and thus, $O(n)_A$ acts non-transitively on the sphere $\mathbb{S}^{n-1}$.

Lemma 4.8. Let $\varepsilon > 0$. For each $D \in M_0(n)$ there exist $A \in \mathcal{R}(n)$ such that $d_H(D, A) < \varepsilon$ and the $O(n)$-stabilizer $O(n)_A$ coincides with the $O(n)$-stabilizer $O(n)_D$.

Proof. According to Theorem 2.2, there is a real $0 < \eta < \varepsilon$ such that if $d_H(C, D) < \eta$ then the stabilizer $O(n)_C$ is conjugate to a subgroup of $O(n)_D$. Let $p_1, \dots, p_k \in D$ be such that the convex hull $P = \operatorname{conv}\big(\{p_1, \dots, p_k\}\big)$ belongs to $M(n)$ (it is enough to chose one of the p_i's lying in $\partial D \cap \mathbb{S}^{n-1}$) and $d_H(D, P) < \eta$. Next, we define

$$A = \operatorname{conv}\big(O(n)_D(p_1) \cup \cdots \cup O(n)_D(p_k)\big).$$

Clearly, $A \in M(n)$ and

$$d_H(D, A) \leq d_H(D, P) < \eta < \varepsilon.$$

Since $O(n)_D$ acts non-transitively on the sphere $\mathbb{S}^{n-1}$, we can apply Lemma 4.4, according to which the boundary ∂A does not contain an $(n-1)$-elliptic domain. In particular, the contact set $\partial A \cap \mathbb{S}^{n-1}$ has empty interior in $\mathbb{S}^{n-1}$, i.e., $A \in \mathcal{R}(n)$.

Because of the choice of η the stabilizer $O(n)_A$ is conjugate to a subgroup of $O(n)_D$. On the other hand, A is an $O(n)_D$-invariant subset, i.e., $O(n)_D \subset O(n)_A$. This implies that $O(n)_A = O(n)_D$, as required. $\square$

The following lemma is just a special case of [8, Theorem 4.5].

Lemma 4.9. Let $X \in cc(\mathbb{R}^n)$ be any convex set. For every $\varepsilon > 0$, the open ball in $cc(\mathbb{R}^n)$ with the radius ε centered at X is convex, i.e., if $\{A_1, \dots, A_k\} \subset cc(\mathbb{R}^n)$ is a finite family such that for every $i = 1, 2, \dots, k$, $d_H(A_i, X) < \varepsilon$, then the set

$$\sum_{i=1}^{k} t_i A_i = \left\{ \sum_{i=1}^{k} t_i a_i \mid a_i \in A_i, \quad i = 1, \dots, k \right\}$$

is ε-close to X, where $t_1, t_2, \dots, t_k \in [0, 1]$ with $\sum_{i=1}^{k} t_i = 1$.

Perhaps, the following is the key result of this section:

Proposition 4.10. For every $\varepsilon > 0$, there is an $O(n)$-map $f_\varepsilon : M_0(n) \to \mathcal{R}(n)$, ε-close to the identity map of $M_0(n)$.

Proof. Let $\mathcal{V} = \{O(X, \varepsilon/4)\}_{X \in M_0(n)}$ be the open cover of $M_0(n)$ consisting of all open balls of radius $\varepsilon/4$. By [7, Lemma 4.1], there exists an $O(n)$-normal cover of $M_0(n)$ (see Section 2 for the definition),

$$\mathcal{W} = \{gS_\mu \mid g \in O(n), \ \mu \in \mathcal{M}\}$$

satisfying the following two conditions:

a) $\mathcal{W}$ is a star-refinement of $\mathcal{V}$. That is to say that for each $gS_\mu \in \mathcal{W}$, there exists an element $V \in \mathcal{V}$ that contains the star of gS_μ with respect to $\mathcal{W}$, i.e.,

$$\mathrm{St}(gS_\mu, \mathcal{W}) = \bigcup \{hS_\lambda \in \mathcal{W} \mid hS_\lambda \cap gS_\mu \neq \emptyset\} \subset V.$$

b) For each $\mu \in \mathcal{M}$, the set S_μ is an H_μ-slice, where H_μ coincides with the stabilizer $O(n)_{X_\mu}$ of a certain point $X_\mu \in S_\mu$.

Since $X_\mu \in M_0(n)$, we see that H_μ acts non-transitively on the sphere $\mathbb{S}^{n-1}$. Thus, by Lemma 4.8, there exists $A_\mu \in \mathcal{R}(n)$ which is $\varepsilon/4$-close to X_μ and $O(n)_{A_\mu} = H_\mu$.

For every $\mu \in \mathcal{M}$, let us denote $O_\mu = O(n)(S_\mu)$. Let $F_\mu : O_\mu \to O(n)(A_\mu)$ be the map defined by

$$F_\mu(gZ) = gA_\mu, \quad Z \in S_\mu, \ g \in O(n).$$

Clearly F_μ is a well-defined continuous $O(n)$-map.

Fix an invariant locally finite partition of unity $\{p_\mu\}_{\mu \in \mathcal{M}}$ subordinated to the open cover $\mathcal{U} = \{O_\mu\}_{\mu \in \mathcal{M}}$, i.e.,

$$\overline{p_\mu^{-1}\big((0, 1]\big)} \subset O_\mu \quad \text{for every} \quad \mu \in \mathcal{M}.$$

Let $\mathcal{N}(\mathcal{U})$ be the nerve of the cover $\mathcal{U}$ and suppose that $\mathcal{M}$ is its vertex set. Denote by $|\mathcal{N}(\mathcal{U})|$ the geometric realization of $\mathcal{N}(\mathcal{U})$. Recall that every point $\alpha \in |\mathcal{N}(\mathcal{U})|$ can be expressed as a sum $\alpha = \sum\limits_{\mu\in\mathcal{M}} \alpha_\mu v_\mu$, where v_μ is the geometric vertex corresponding to $\mu \in \mathcal{M}$ and α_μ, $\mu \in \mathcal{M}$ are the baricentric coordinates of α.

For a simplex σ of the nerve $\mathcal{N}(\mathcal{U})$ with the vertices $\mu_0, \dots, \mu_k$, we will use the notation $\sigma = \langle \mu_0, \dots, \mu_k\rangle$. By $|\langle \mu_0, \dots \mu_k\rangle|$ we denote the corresponding geometric simplex with the geometric vertices $v_{\mu_0}, \dots, v_{\mu_k}$.

For every geometric simplex $|\sigma| = |\langle \mu_0, \dots \mu_k\rangle| \subset |\mathcal{N}(\mathcal{U})|$, let us denote by $\beta(\sigma) \in |\mathcal{N}(\mathcal{U})|$ the geometric baricenter of $|\sigma|$, i.e., $\beta(\sigma) = \sum\limits_{\mu\in\mathcal{M}} \beta(\sigma)_\mu v_\mu$, where

$$\beta(\sigma)_\mu = \begin{cases} 1/k+1, & \text{if } \mu \in \{\mu_0, \dots, \mu_k\}, \\ 0, & \text{if } \mu \notin \{\mu_0, \dots, \mu_k\}. \end{cases}$$

Let us consider the map $\Psi : |\mathcal{N}(\mathcal{U})| \to |\mathcal{N}(\mathcal{U})|$ defined in each $\alpha = \sum\limits_{\mu\in\mathcal{M}} \alpha_\mu v_\mu \in |\mathcal{N}(\mathcal{U})|$ as follows: if $|\langle \mu_0, \dots \mu_k\rangle|$ is the carrier of α and $\alpha_{\mu_0} \geq \alpha_{\mu_1} \geq \cdots \geq \alpha_{\mu_k}$, then

$$\Psi(\alpha) = \sum_{\sigma\in\mathcal{N}(\mathcal{U})} \Psi(\alpha)_\sigma \beta(\sigma)$$

where

(4.4)
$$\Psi(\alpha)_\sigma = \begin{cases} (i+1)\big(\alpha_{\mu_i} - \alpha_{\mu_{i+1}}\big), & \text{if } \sigma = \langle \mu_0, \dots, \mu_i\rangle,\ i = 0, \dots, k-1, \\ (k+1)\alpha_{\mu_k}, & \text{if } \sigma = \langle \mu_0, \dots, \mu_k\rangle, \\ 0, & \text{if } \sigma \neq \langle \mu_0, \dots, \mu_i\rangle,\ i = 0, \dots, k. \end{cases}$$

It is not difficult to see that Ψ is the identity map of $|\mathcal{N}(\mathcal{U})|$ written in the baricentric coordinates with respect to the first baricentric subdivision of $|\mathcal{N}(\mathcal{U})|$; we shall need this representation in the sequel.

Let $p : M_0(n) \to |\mathcal{N}(\mathcal{U})|$ be the canonical map defined by

$$p(X) = \sum_{\mu\in\mathcal{M}} p_\mu(X) v_\mu, \quad X \in M_0(n).$$

Since each p_μ is $O(n)$-invariant, the map p is also $O(n)$-invariant.

For every simplex $\sigma = \langle \mu_0, \dots, \mu_k\rangle \in \mathcal{N}(\mathcal{U})$ the set $V_\sigma = O_{\mu_0} \cap \cdots \cap O_{\mu_k}$ is a nonempty open subset of $M_0(n)$. Continuity of the union operator and the convex hull operator (see, e.g., [28, Corollary 5.3.7] and [29, Theorem 2.7.4 (iv)]) imply that the map $\Omega'_\sigma : V_\sigma \to M_0(n)$ given by

$$\Omega'_\sigma(X) = \operatorname{conv}\left(\bigcup_{\mu\in\sigma} F_\mu(X)\right), \quad X \in V_\sigma,$$

is a continuous $O(n)$-map.

Observe that $\Omega'_\sigma(X)$ belongs to $M_0(n)$ and the contact set $\Omega'_\sigma(X)\cap\mathbb{S}^{n-1}$ is contained in the contact set $\big(\bigcup_{\mu\in\sigma} F_\mu(X)\big)\cap\mathbb{S}^{n-1} = \bigcup_{\mu\in\sigma}(F_\mu(X)\cap\mathbb{S}^{n-1})$, and hence,

$$\Omega'_\sigma(X)\cap\mathbb{S}^{n-1} \quad \text{has empty interior in} \quad \mathbb{S}^{n-1}. \tag{4.5}$$

Fix a set $B \in M_0(n)$. For each simplex σ of $\mathcal{N}(\mathcal{U})$, we extend the map Ω'_σ to a function $\Omega_\sigma : M_0(n) \to M_0(n)$ as follows:

$$\Omega_\sigma(X) = \begin{cases} \Omega'_\sigma(X) & \text{if } X \in V_\sigma, \\ B, & \text{if } X \notin V_\sigma. \end{cases}$$

The desired map $f_\varepsilon : M_0(n) \to M_0(n)$ can now be defined by the formula:

$$f_\varepsilon(X) = \sum_{\sigma\in\mathcal{N}(\mathcal{U})} \Psi\big(p(X)\big)_\sigma\Omega_\sigma(X), \qquad X \in M_0(n).$$

For every $X \in M_0(n)$, let $Q(X)$ be the subset of $\mathcal{M}$ consisting of all $\mu \in \mathcal{M}$ such that $X \in p_\mu^{-1}\big((0,1]\big)$. Similarly, denote by $Q'(X)$ the subset of $\mathcal{M}$ consisting of all $\mu \in \mathcal{M}$ such that $X \in \overline{p_\mu^{-1}\big((0,1]\big)}$.

It is clear that $Q(X) \subset Q'(X)$ and, due to local finiteness of the cover $\{\overline{p_\mu^{-1}\big((0,1]\big)}\}_{\mu\in\mathcal{M}}$, both sets are finite. Moreover, it follows from the formula (4.4) that $\Psi\big(p(X)\big)_\sigma = 0$ whenever $\sigma \not\subset Q'(X)$.

Then, for every $X \in M_0(n)$ we have:

$$f_\varepsilon(X) = \sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi\big(p(X)\big)_\sigma\Omega_\sigma(X) = \sum_{\substack{\sigma\mathcal{N}(\mathcal{U})\\ \sigma\subset Q'(X)}} \Psi\big(p(X)\big)_\sigma\Omega_\sigma(X). \tag{4.6}$$

To see the continuity of f_ε, let us fix an arbitrary point $C \in M_0(n)$. Define

$$V = \Big(\bigcap_{\mu\in Q'(C)} O_\mu\Big) \setminus \bigcup_{\mu\notin Q'(C)} \overline{p_\mu^{-1}\big((0,1]\big)}.$$

Since the family $\{p_\mu^{-1}((0,1])\}_{\mu\in\mathcal{M}}$ is locally finite, the union $\bigcup_{\mu\notin Q'(C)} \overline{p_\mu^{-1}\big((0,1]\big)}$ is closed, and therefore, V is a neighborhood of C. It is evident that for every $X \in V$, the set $Q(X)$ is contained in $Q'(C)$. Using equality (4.6), we infer that

$$f_\varepsilon(X) = \sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q'(C)}} \Psi\big(p(X)\big)_\sigma\Omega_\sigma(X) \quad \text{for every} \quad X \in V.$$

Observe that $V \subset V_\sigma$ for every simplex $\sigma \in \mathcal{N}(\mathcal{U})$ such that $\sigma \subset Q'(C)$, and hence, the restriction $\Omega_\sigma|_V = \Omega'_\sigma|_V$ is continuous in V.

On the other hand, $\Psi\big(p(X)\big)_\sigma$ is just the $\beta(\sigma)$-th baricentric coordinate of $\Psi(p(X))$. Thus, for every $\sigma \in \mathcal{N}(\mathcal{U})$, the map $X \mapsto \Psi(p(X))_\sigma$ depends continuously on X. So, $f_\varepsilon|_V$ is a finite sum of continuous functions and therefore it is also continuous in V. Consequently, f_ε is continuous at the point $C \in M_0(n)$, as required.

If $g \in O(n)$ and $X \in M_0(n)$, then

$$\begin{aligned} f_\varepsilon(gX) &= \sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi\big(p(gX)\big)_\sigma \Omega_\sigma(gX) = \sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi\big(p(X)\big)_\sigma \Omega'_\sigma(gX) \\ &= \sum_{\sigma\in\mathcal{N}(\mathcal{U})} \Psi\big(p(X)\big)_\sigma \big(g\Omega'_\sigma(X)\big) = g\Big(\sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi\big(p(X)\big)_\sigma \Omega'_\sigma(X)\Big) \\ &= g\Big(\sum_{\substack{\sigma\in\mathcal{N}(\mathcal{U})\\ \sigma\subset Q(X)}} \Psi\big(p(X)\big)_\sigma \Omega_\sigma(X)\Big) = gf_\varepsilon(X), \end{aligned}$$

which shows that f_ε is $O(n)$-equivariant.

To see that $f_\varepsilon(X)$ belongs to $M_0(n)$, let us suppose that

$$Q(X) = \{\mu_0, \dots, \mu_k\} \quad \text{and} \quad p_{\mu_0}(X) \geq p_{\mu_1}(X) \geq \dots \geq p_{\mu_k}(X).$$

Then, according to equalities (4.4) and (4.6), the set $f_\varepsilon(X)$ can be seen as the convex sum:

$$\begin{aligned} f_\varepsilon(X) &= (k+1)p_{\mu_k}(X)\Omega_{\langle\mu_0,\dots,\mu_k\rangle}(X) \\ &\quad + \sum_{i=0}^{k-1}(i+1)\Big(p_{\mu_i}(X) - p_{\mu_{i+1}}(X)\Big)\Omega_{\langle\mu_0,\dots,\mu_i\rangle}(X) \\ &= (k+1)p_{\mu_k}(X)\Omega'_{\langle\mu_0,\dots,\mu_k\rangle}(X) \\ &\quad + \sum_{i=0}^{k-1}(i+1)\Big(p_{\mu_i}(X) - p_{\mu_{i+1}}(X)\Big)\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X). \end{aligned}$$

Thus, $f_\varepsilon(X)$ is a convex subset contained in $\mathbb{B}^n$. Furthermore, observe that $F_{\mu_0}(X) \subset \Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X)$ for every $i = 0, \dots, k$. This implies that

$$\begin{aligned} F_{\mu_0}(X) &= (k+1)p_{\mu_k}(X)F_{\mu_0}(X) + \sum_{i=0}^{k-1}(i+1)\Big(p_{\mu_i}(X) - p_{\mu_{i+1}}(X)\Big)F_{\mu_0}(X) \\ &\subset (k+1)p_{\mu_k}(X)\Omega'_{\langle\mu_0,\dots,\mu_k\rangle}(X) \\ &\quad + \sum_{i=0}^{k-1}(i+1)\Big(p_{\mu_i}(X) - p_{\mu_{i+1}}(X)\Big)\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X) \\ &= f_\varepsilon(X). \end{aligned}$$

Since $F_{\mu_0}(X) \in M_0(n)$, the inclusion $F_{\mu_0}(X) \subset f_\varepsilon(X)$ yields that $f_\varepsilon(X) \in M_0(n)$.

On the other hand, the contact set $f_\varepsilon(X)\cap\mathbb{S}^{n-1}$ is contained in

$$\Big(\bigcup_{i=0}^{k}\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X)\Big)\cap\mathbb{S}^{n-1}=\bigcup_{i=0}^{k}\Big(\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X)\cap\mathbb{S}^{n-1}\Big).$$

Further, since by (4.5), each $\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X)\cap\mathbb{S}^{n-1}$ has empty interior in $\mathbb{S}^{n-1}$, we infer that the finite union $\bigcup_{i=0}^{k}\Big(\Omega'_{\langle\mu_0,\dots,\mu_i\rangle}(X)\cap\mathbb{S}^{n-1}\Big)$ also has empty interior in $\mathbb{S}^{n-1}$. This yields that $f_\varepsilon(X)\cap\mathbb{S}^{n-1}$ has empty interior in $\mathbb{S}^{n-1}$, as required.

It remains only to prove that $d_H\big(X,f_\varepsilon(X)\big)<\varepsilon$ for every $X\in M_0(n)$.

Since $f_\varepsilon(X)$ is a convex sum of the sets $\Omega_{\langle\mu_0,\dots,\mu_i\rangle}(X)$ for $i=0,\dots,k$, according to Lemma 4.9, it is enough to prove that $\Omega_{\langle\mu_0,\dots,\mu_i\rangle}(X)$ is ε-close to X for every $i=0,\dots,k$.

Recall that $\Omega_{\langle\mu_0,\dots,\mu_i\rangle}(X)=\operatorname{conv}\big(\bigcup_{j=0}^{i}F_{\mu_j}(X)\big)$, and hence, we only have to prove that each $F_{\mu_j}(X)$ satisfies $d_H\big(X,F_{\mu_j}(X)\big)<\varepsilon$.

For this purpose, suppose that $g_j\in O(n)$ is such that $F_{\mu_j}(X)=g_jA_{\mu_j}$. Then $X\in g_jS_{\mu_j}$ and $g_jX_{\mu_j}\in g_jS_{\mu_j}$.

Since $\mathcal{W}$ is a star-refinement of $\mathcal{V}$, there exists a point $Z\in M_0(n)$ such that the star $St(X,\mathcal{W})=\bigcup\{gS_\mu\in\mathcal{W}\mid X\in gS_\mu\}$ is contained in $O(Z,\varepsilon/4)$. In particular,

$$d_H(X,Z)<\varepsilon/4\quad\text{and}\quad d_H(g_jX_{\mu_j},Z)<\varepsilon/4. \tag{4.7}$$

This implies that $d_H(g_jX_{\mu_j},X)<\varepsilon/2$. By the choice of A_{μ_j}, we have that $d_H(A_{\mu_j},X_{\mu_j})<\varepsilon/4$. Since the Hausdorff metric is $O(n)$-invariant we get

$$d_H(g_jA_{\mu_j},g_jX_{\mu_j})=d_H(A_{\mu_j},X_{\mu_j})<\varepsilon/4,$$

and hence,

$$\begin{aligned}d_H(X,F_{\mu_j}(X))&=d_H(X,g_jA_{\mu_j})\\&\le d_H(X,g_jX_{\mu_j})+d_H(g_jX_{\mu_j},g_jA_{\mu_j})\\&<\varepsilon/2+\varepsilon/4<\varepsilon,\end{aligned}$$

as required. □

Proposition 4.11. For every $\varepsilon>0$, there is an $O(n)$-map, $h_\varepsilon: M_0(n)\to M_0(n)\setminus\mathcal{R}(n)$, ε-close to the identity map of $M_0(n)$.

Proof. Define a continuous map $\gamma: M_0(n)\to\mathbb{R}$ by the rule:

$$\gamma(A)=\frac{1}{2}\min\{\varepsilon,d_H(\mathbb{B}^n,A)\},\quad\text{for every } A\in M_0(n).$$

Let $h_\varepsilon(A)$ be just the closed $\gamma(A)$-neighborhood of A in $\mathbb{B}^n$, i.e.,

$$h_\varepsilon(A) = A_{\gamma(A)} = \{x \in \mathbb{B}^n \mid d(x, A) \leq \gamma(A)\},\ A \in M_0(n).$$

By the choice of $\gamma(A)$, the set $h_\varepsilon(A)$ is different from $\mathbb{B}^n$, and since $A \subset h_\varepsilon(A)$, we see that $h_\varepsilon(A) \in M_0(n)$. Even more, $h_\varepsilon(A) \cap \mathbb{S}^{n-1}$ has nonempty interior in the unit sphere $\mathbb{S}^{n-1}$. Thus, $h_\varepsilon(A) \in M_0(n) \setminus \mathcal{R}(n)$.

By [7, Lemma 5.3], $d_H(A, A_{\gamma(A)}) < \gamma_A < \varepsilon$ which implies that h_ε is ε-close to the identity map of $M_0(n)$.

Let us check the continuity of h_ε. For any $A, C \in M_0(n)$ the following inequality holds:

$$d_H(h_\varepsilon(A), h_\varepsilon(C)) = d_H(A_{\gamma(A)}, C_{\gamma(C)}) \leq d_H(A_{\gamma(A)}, A_{\gamma(C)}) + d_H(A_{\gamma(C)}, C_{\gamma(C)}).$$

But,

$$d_H(A_{\gamma(A)}, A_{\gamma(C)}) \leq |\gamma(A) - \gamma(C)| \quad \text{and} \quad d_H(A_{\gamma(C)}, C_{\gamma(C)}) \leq d_H(A, C)$$

(see, e.g., [7, Lemma 5.3]).

Consequently, we get:

$$d_H(h_\varepsilon(A), h_\varepsilon(C)) \leq |\gamma(A) - \gamma(C)| + d_H(A, C).$$

Now the continuity of γ implies the one of h_ε.

□

As a consequence of Propositions 4.10 and 4.11 we have the following corollaries.

Corollary 4.12. For any closed subgroup $K \subset O(n)$, the K-orbit space $M_0(n)/K$ is a Q-manifold.

Proof. Consider the metric on $M_0(n)/K$ induced by d_H according to equality (2.1).

Clearly, $M_0(n)$ is a locally compact space, and thus, the orbit space $M_0(n)/K$ is also locally compact. Since $M(n)$ is an $O(n)$-AR, and $M_0(n)$ is an open $O(n)$-invariant set in $M(n)$, we infer that $M_0(n)$ is an $O(n)$-ANR. This in turn implies that $M_0(n)$ is a K-ANR (see, e.g., [27]). Then, by Theorem 2.3, the orbit space $M_0(n)/K$ is an ANR.

According to Toruńczyk's Characterization Theorem [26, Theorem 1], it remains to check that for every $\varepsilon > 0$, there exist continuous maps $\widetilde{f}_\varepsilon, \widetilde{h}_\varepsilon : M_0(n)/K \to M_0(n)/K$, ε-close to the identity map of $M_0(n)/K$ such that the images $\operatorname{Im} \widetilde{f}_\varepsilon$ and $\operatorname{Im} \widetilde{h}_\varepsilon$ are disjoint.

Let f_ε and h_ε be the $O(n)$-maps from Propositions 4.10 and 4.11, respectively. They induce continuous maps $\widetilde{f}_\varepsilon : M_0(n)/K \to M_0(n)/K$ and

$\widetilde{h}_\varepsilon : M_0(n)/K \to M_0(n)/K$. Since $\operatorname{Im}\widetilde{f}_\varepsilon = (\operatorname{Im} f_\varepsilon)/K$, $\operatorname{Im}\widetilde{h}_\varepsilon = (\operatorname{Im} h_\varepsilon)/K$ and $\operatorname{Im} f_\varepsilon \cap \operatorname{Im} h_\varepsilon = \emptyset$, we infer that $\operatorname{Im}\widetilde{f}_\varepsilon \cap \operatorname{Im}\widetilde{h}_\varepsilon = \emptyset$.

On the other hand, since f_ε and h_ε are ε-close to the identity map of $M_0(n)$, using inequality (2.2), we get that $\widetilde{f}_\varepsilon$ and $\widetilde{h}_\varepsilon$ are ε-close to the identity map of $M_0(n)/K$. This completes the proof.

□

Corollary 4.13. For any closed subgroup $K \subset O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, the K-orbit space $M(n)/K$ is a Hilbert cube. In particular, $M(n)$ is homeomorphic to the Hilbert cube.

Proof. We have already seen in Corollary 4.7 that the singleton $\{\mathbb{B}^n\}$ is a Z-set in $M(n)/K$. Observe that the Q-manifold $M_0(n)/K$ can be seen as the complement $\big(M(n)/K\big) \setminus \{\mathbb{B}^n\}$. It then follows from [26, §3] that $M(n)/K$ is also a Q-manifold. Furthermore, $M(n)/K$ is compact and contractible. But since the only compact contractible Q-manifold is the Hilbert cube (see [28, Theorem 7.5.8]), we conclude that $M(n)/K$ is homeomorphic to the Hilbert cube. □

Corollary 4.14. For any closed subgroup $K \subset O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, the K-fixed point set $M(n)^K$ is homeomorphic to the Hilbert cube.

Proof. Since $M(n)$ is compact and $M(n)^K$ is closed in $M(n)$, we see that $M(n)^K$ is also compact. By Theorem 4.3, $M(n)$ is an $O(n)$-AR. This, in combination with [9, Theorem 3.7], yields that $M(n)^K$ is an AR. In particular, $M(n)^K$ is contractible.

Let f_ε and h_ε be the $O(n)$-maps from Propositions 4.10 and 4.11, respectively. Due to equivariance, we have

$$f_\varepsilon\big(M_0(n)^K\big) \subset M_0(n)^K \quad \text{and} \quad h_\varepsilon\big(M_0(n)^K\big) \subset M_0(n)^K.$$

By virtue of Toruńczyk's Characterization Theorem [26, Theorem 1], we conclude that $M_0(n)^K$ is a Q-manifold. But $M_0(n)^K = M(n)^K \setminus \{\mathbb{B}^n\}$ and Corollary 4.7 implies that the singleton $\{\mathbb{B}^n\}$ is a Z-set in $M(n)^K$. This yields that $M(n)^K$ is also a Q-manifold (see [26, §3]). Furthermore, $M(n)^K$ is compact and contractible. Since the only compact contractible Q-manifold is the Hilbert cube (see [28, Theorem 7.5.8]), we conclude that $M(n)^K$ is homeomorphic to the Hilbert cube. □

We resume all the above results about the $O(n)$-space $M(n)$ in the following corollary:

Corollary 4.15. $M(n)$ is a Hilbert cube endowed with an $O(n)$-action satisfying the following properties:

(1) $M(n)$ is an $O(n)$-AR with a unique $O(n)$-fixed point, $\mathbb{B}^n$,
(2) $M(n)$ is strictly $O(n)$-contractible to $\mathbb{B}^n$,
(3) For a closed subgroup $K \subset O(n)$, the set $M(n)^K$ equals the singleton $\{\mathbb{B}^n\}$ if and only if K acts transitively on the unit sphere $\mathbb{S}^{n-1}$, and $M(n)^K$ is homeomorphic to the Hilbert cube whenever $M(n)^K \neq \{\mathbb{B}^n\}$,
(4) For any closed subgroup $K \subset O(n)$, the K-orbit space $M_0(n)/K$ is a Q-manifold.

This corollary in combination with [10, Theorem 3.3], yields the following:

Theorem 4.16. The orbit space $M(n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$.

5. Some properties of $L(n)$

Recall that $L(n)$ is the hyperspace of all compact convex bodies for which the Euclidean unit ball is the minimum-volume ellipsoid of Löwner.

In [7] the subset $L'(n)$ of $L(n)$ consisting of all $A \in L(n)$ with $A = -A$ was studied. It turns out that $L(n)$ enjoys all the properties of $L'(n)$ established in [7], and an easy modification of the method developed in [7, section 5] allows one to establish similar properties of $L(n)$. However, seeking for completeness, we shall provide in this section some more specific details and appropriate new references.

Proposition 5.1. $L(n)$ is an $O(n)$-AR.

Proof. It was proved in [8, Corollary 4.8] that $cb(\mathbb{R}^n)$ is an $O(n)$-AR. Since $L(n)$ is a global $O(n)$-slice in $cb(\mathbb{R}^n)$, according to Corollary 3.9(2), there exists an $O(n)$-equivariant retraction $r : cb(\mathbb{R}^n) \to L(n)$. This yields that $L(n)$ is also an $O(n)$-AR. $\square$

Proposition 5.2. The map $F : L(n) \times [0,1] \to L(n)$ defined by

$$F(A,t) = (1-t)A + t\mathbb{B}^n$$

is an $O(n)$-strict contraction such that $F(A,1) = \mathbb{B}^n$. In particular, for every closed subgroup $K \subset O(n)$, the orbit space $L(n)/K$ is contractible to its point $\mathbb{B}^n$.

Proof. It is evident that F satisfies the first condition of the proposition. Letting $\widetilde{F}\big(K(A),t\big) = K\big(F(A,t)\big)$ we obtain a deformation of $L(n)/K$ to the point $\mathbb{B}^n \in L(n)/K$, thus proving that $L(n)/K$ is contractible. □

By $\mathcal{P}(n)$ we will denote the subset of $L(n)$ consisting of all compact convex bodies $A \in L(n)$ such that the contact set $A \cap \partial\mathbb{B}^n$ has empty interior in the boundary sphere $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$.

Denote by $L_0(n)$ the complement $L(n) \setminus \{\mathbb{B}^n\}$.

Lemma 5.3. Let $\varepsilon > 0$. For each convex body $X \in L_0(n)$, there exists a convex body $A \in \mathcal{P}(n)$ such that $d_H(X,A) < \varepsilon$ and the $O(n)$-stabilizer $O(n)_A$ coincides with the $O(n)$-stabilizer $O(n)_X$.

Although the proof of Lemma 5.3 is similar to the one of Lemma 4.8, there is a significant difference, and for this reason we shall present the complete proof here.

Proof. Let $r : cb(\mathbb{R}^n) \to L(n)$ be the $O(n)$-equivariant retraction used in the proof of Proposition 5.1 (c.f. Corollary 3.9(2)). According to Theorem 2.2, there is a $O(n)_X$-slice S such that $X \in S$ and $[O(n)_C] \preceq [O(n)_X]$ whenever $C \in O(n)(S)$. Since $O(n)(S)$ is open, there exist a number $0 < \eta < \varepsilon$ such that $O(X,\eta) \subset O(n)(S)$. In particular, if $C \in O(X,\eta)$ then $[O(n)_C] \preceq [O(n)_X]$.

Since $L(n)$ is compact, there exists $0 < \delta < \eta/2$ such that $d_H(r(C),C) < \eta/2$ for every C lying in the δ-neighborhood of $L(n)$.

Let $p_1,\dots,p_k \in \partial X$ be such that the convex hull $P = \operatorname{conv}\big(\{p_1,\dots,p_k\}\big)$ has nonempty interior in $\mathbb{R}^n$ and $d_H(P,X) < \delta$. Consider the convex hull

$$D = \operatorname{conv}\big(O(n)_X(p_1) \cup \dots \cup O(n)_X(p_k)\big).$$

Since $P \subset D$, we see that D has nonempty interior, and hence, $D \in cb(\mathbb{R}^n)$. Since $O(n)_X$ acts non-transitively on $\mathbb{S}^{n-1}$, we can apply Lemma 4.4, according to which the boundary ∂D does not contain an $(n-1)$-elliptic domain. In particular, the contact set $D \cap \partial l(D)$ does not contain an elliptic domain (recall that here $l(D)$ denotes the minimal volume ellipsoid containing D).

Let $A = r(D)$. Since $A \in L(n)$ and A lies in the $\operatorname{Aff}(n)$-orbit of D (see Corollary 3.9(1)), there exists an affine transformation g such that $A = gD$. The contact set $A \cap \mathbb{S}^{n-1}$ is the image under g of the contact set $D \cap \partial l(D)$, and thus, it has empty interior in the sphere $\mathbb{S}^{n-1}$. Hence, A belongs to $\mathcal{P}(n)$. The construction of P guarantees that $P \subset D \subset X$, and therefore,

$$d_H(D,X) \le d_H(P,X) < \delta < \eta/2.$$

By the choice of δ one has $d_H(r(D), D) < \eta/2$, and hence,

$$\begin{aligned} d_H(A, X) &\leq d_H(A, D) + d_H(D, X) \\ &= d_H(r(D), D) + d_H(D, X) < \eta/2 + \eta/2 = \eta. \end{aligned}$$

Thus, $d_H(A, X) < \eta < \varepsilon$, as required.

Furthermore, due to the choice of η, $O(n)_A$ is conjugate to a subgroup of $O(n)_X$. It remains to prove that $O(n)_X = O(n)_A$. Since D is an $O(n)_X$-invariant subset, one has $O(n)_X \subset O(n)_D$. Also, since r is an $O(n)$-map, we have

$$O(n)_D \subset O(n)_{r(D)} = O(n)_A.$$

Thus, $O(n)_X \subset O(n)_A$ which implies, in combination with $[O(n)_A] \preceq [O(n)_X]$, that $O(n)_A = O(n)_X$, as required. □

Proposition 5.4. For every $\varepsilon > 0$, there is an $O(n)$-map, $f_\varepsilon : L_0(n) \to \mathcal{P}(n)$, ε-close to the identity map of $L_0(n)$.

Proof. Repeat the proof of Proposition 4.10, replacing $M_0(n)$ by $L_0(n)$, as far as the construction of the family $\{X_\mu\}_{\mu\in\mathcal{M}}$. Next, use Lemma 5.3 to find, for every index μ, a compact set A_μ, $\varepsilon/4$-close to X_μ such that $O(n)_{A_\mu} = H_\mu$.

Now the proof follows by repeating the rest of the proof of Proposition 4.10, previously replacing $M_0(n)$ by $L_0(n)$, and $\mathcal{R}(n)$ by $\mathcal{P}(n)$. □

Proposition 5.5. For every $\varepsilon > 0$, there is an $O(n)$-map, $h_\varepsilon : L_0(n) \to L_0(n) \setminus \mathcal{P}(n)$, ε-close to the identity map of $L(n)$ such that $h_\varepsilon(A) \neq \mathbb{B}^n$ for every $A \in L(n)$.

Proof. The proof follows by repeating the proof of Proposition 4.11, previously replacing $M_0(n)$ by $L_0(n)$, and $M_0(n) \setminus \mathcal{R}(n)$ by $L_0(n) \setminus \mathcal{P}(n)$.

□

Proposition 5.6. Let $K \subset O(n)$ be a a closed subgroup that acts non-transitively on $\mathbb{S}^{n-1}$. Then, for every $\varepsilon > 0$, there exists a K-equivariant map $\chi_\varepsilon : L(n) \to L_0(n)$, ε-close to the identity map of $L(n)$.

Proof. The proof goes as the one of Proposition 4.6, if we replace $M(n)$ by $L(n)$, $M_0(n)$ by $L_0(n)$, $cc(\mathbb{R}^n)$ by $cb(\mathbb{R}^n)$, and the retraction r of (4.2) by the retraction $r : cb(\mathbb{R}^n) \to L(n)$ given in Corollary 3.9(2). We omit the details. □

In the same manner that Proposition 4.6 implies Corollary 4.7, we infer from Proposition 5.6 the following corollary:

Corollary 5.7. For every closed subgroup $K \subset O(n)$ that acts non transitively on the unit sphere $\mathbb{S}^{n-1}$,

(1) the singleton $\{\mathbb{B}^n\}$ is a Z-set in $L(n)^K$,
(2) the class of $\{\mathbb{B}^n\}$ is a Z-set in $L(n)/K$.

Proposition 5.8. For every closed subgroup $K \subset O(n)$, $L_0(n)/K$ is a Q-manifold.

Proof. By Proposition 5.1, $L(n)$ is an $O(n)$-AR, which in turn implies that $L(n) \in K$-AR (see, e.g., [27]). Then, Theorem 2.3 implies that $L(n)/K$ is an AR. Since $L_0(n)/K$ is open in $L(n)/K$ we conclude that $L_0(n)/K$ is a locally compact ANR.

According to Toruńczyk's Characterization Theorem [26, Theorem 1], it is enough to check that for every $\varepsilon > 0$, there exist continuous maps $\widetilde{f}_\varepsilon, \widetilde{h}_\varepsilon : L_0(n)/K \to L_0(n)/K$ ε-close to the identity map of $L_0(n)/K$ such that $\operatorname{Im} \widetilde{f}_\varepsilon \cap \operatorname{Im} \widetilde{h}_\varepsilon = \emptyset$.

Let f_ε and h_ε be the $O(n)$-maps constructed in Propositions 5.4 and 5.5, respectively. They induce continuous maps $\widetilde{f}_\varepsilon : L_0(n)K \to L_0(n)/K$ and $\widetilde{h}_\varepsilon : L_0(n)/K \to L_0(n)/K$. Since $\operatorname{Im} \widetilde{f}_\varepsilon = (\operatorname{Im} f_\varepsilon)/K$, $\operatorname{Im} \widetilde{h}_\varepsilon = (\operatorname{Im} h_\varepsilon)/K$ and $\operatorname{Im} f_\varepsilon \cap \operatorname{Im} h_\varepsilon = \emptyset$, we infer that $\operatorname{Im} \widetilde{f}_\varepsilon \cap \operatorname{Im} \widetilde{h}_\varepsilon = \emptyset$. Since f_ε and h_ε are ε-close to the identity map of $L_0(n)$, using inequality (2.2), we get that $\widetilde{f}_\varepsilon$ and $\widetilde{h}_\varepsilon$ are ε-close to the identity map of $L_0(n)/K$, as required. $\square$

Now, Proposition 5.8 , Corollary 5.7 and [26, §3] imply that $L(n)/K$ is a Q-manifold if $K \subset O(n)$ is a closed subgroup that acts non-transitively on the sphere $\mathbb{S}^{n-1}$. Since $L(n)/K$ is compact and contractible, we infer from [28, Theorem 7.5.8] the following corollary:

Corollary 5.9. For every closed subgroup $K \subset O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, the K-orbit space $L(n)/K$ is a Hilbert cube. In particular, $L(n)$ is a Hilbert cube.

Repeating the same steps used in the proof of Corollary 4.14, we can infer from Corollary 5.7 and Propositions 5.4 and 5.5 the following result:

Corollary 5.10. For any closed subgroup $K \subset O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, the K-fixed point set $L(n)^K$ is homeomorphic to the Hilbert cube.

Finally, likewise to the case of $M(n)$, we can infer from all previous results of this section that $L(n)$ is a Hilbert cube endowed with an $O(n)$-action that satisfies the following conditions:

(1) $L(n)$ is an $O(n)$-AR with a unique $O(n)$-fixed point, $\mathbb{B}^n$,
(2) $L(n)$ is strictly $O(n)$-contractible to $\mathbb{B}^n$,
(3) For a closed subgroup $K \subset O(n)$, the set $L(n)^K$ equals the singleton $\{\mathbb{B}^n\}$ if and only if K acts transitively on the unit sphere $\mathbb{S}^{n-1}$, and $L(n)^K$ is homeomorphic to the Hilbert cube whenever $L(n)^K \neq \{\mathbb{B}^n\}$,
(4) For any closed subgroup $K \subset O(n)$, the K-orbit space $L_0(n)/K$ is a Q-manifold.

These properties in combination with [10, Theorem 3.3], yield the following:

Theorem 5.11. The orbit space $L(n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$.

6. Orbit spaces of $cb(\mathbb{R}^n)$

In what follows we will denote by $cb_0(\mathbb{R}^n)$ the complement:

$$cb_0(\mathbb{R}^n) = cb(\mathbb{R}^n) \setminus E(n).$$

In this section we shall prove the following main result:

Theorem 6.1. Let $K \subset O(n)$ be a closed subgroup that acts non-transitively on $\mathbb{S}^{n-1}$. Then:

(1) the orbit space $cb_0(\mathbb{R}^n)/K$ is a Q-manifold.
(2) the orbit space $cb(\mathbb{R}^n)/K$ is a Q-manifold homeomorphic to $(E(n)/K) \times Q$.

By Corollary 3.9(2) we have an $O(n)$-equivariant homeomorphism

$$cb(\mathbb{R}^n) \cong_{O(n)} L(n) \times E(n).$$

Under this homeomorphism, $cb_0(\mathbb{R}^n)$ corresponds to the product $E(n) \times L_0(n)$, thus we have the following $O(n)$-equivariant homeomorphism:

$$cb(\mathbb{R}^n) \cong_{O(n)} L(n) \times E(n). \tag{6.1}$$

In the sequel we will consider the following $O(n)$-invariant metric on the product $E(n) \times L(n)$:

$$D\big((A_1, E_1), (A_2, E_2)\big) = d_H(A_1, A_2) + d_H(E_1, E_2).$$

Proposition 6.2. For each $\varepsilon > 0$ and every closed subgroup $K \subset O(n)$ that acts non-transitively on $\mathbb{S}^{n-1}$, there exists a K-equivariant map $\eta : cb(\mathbb{R}^n) \to cb_0(\mathbb{R}^n)$ which is ε-close to the identity map of $cb(\mathbb{R}^n)$.

Proof. Let $\varepsilon > 0$. By Proposition 5.6, there exists a K-map, $\chi_\varepsilon : L(n) \to L_0(n)$, such that $d_H\big(A, \xi(A)\big) < \varepsilon$ for every $A \in L(n)$. Then, the map

$$\eta = \chi_\varepsilon \times Id : L(n) \times E(n) \to L_0(n) \times E(n)$$

is a K-map such that

$$D\big(\eta(A, E), (A, E)\big) = d_H\big(\xi(A), A\big) < \varepsilon.$$

$\square$

The map η of Proposition 6.2 induces a map

$$\widetilde{\eta}\colon \frac{L(n) \times E(n)}{K} \longrightarrow \frac{L_0(n) \times E(n)}{K}$$

which, by virtue of inequality (2.2), is ε-close to the identity map of $\frac{L(n)\times E(n)}{K}$. This yields the following corollary:

Corollary 6.3. For every closed subgroup $K \subset O(n)$ that acts non-transitively on $\mathbb{S}^{n-1}$, $E(n)/K$ is a Z-set in $cb(\mathbb{R}^n)/K$. In particular, $E(n)$ is a Z-set in $cb(\mathbb{R}^n)$.

Proposition 6.4. Let $K \subset O(n)$ be a closed subgroup that acts non-transitively on $\mathbb{S}^{n-1}$ and $\pi : L(n) \times E(n) \to E(n)$ be the second projection. Then the induced map $\widetilde{\pi} : \big(L(n) \times E(n)\big)/K \to E(n)/K$ is proper and has contractible fibers.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} L(n) \times E(n) \arrow[r, "\pi"] \arrow[d, "p_1"'] & E(n) \arrow[d, "p_2"] \\ \frac{L(n)\times E(n)}{K} \arrow[r, "\widetilde{\pi}"'] & \frac{E(n)}{K}, \end{tikzcd}$$

where p_1 and p_2 are the respective K-orbit maps.

Properness of $\widetilde{\pi}$ easily follows from compactness of $L(n)$ and K. That the fibers of $\widetilde{\pi}$ are contractible follows immediately from the fact that $L(n)$ is $O(n)$-equivariantly contractible (see Proposition 5.2). $\square$

Theorem 6.5 (R. D. Edwards)**.** Let M be a Q manifold and Y a locally compact ANR. If there exists a CE-map $f : M \to Y$, then M is homeomorphic to $Y \times Q$.

Proof. Since f is a CE-map, then by a theorem of R. D. Edwards [14, Theorem 43.1], the product map

$$f \times Id : M \times Q \to Y \times Q$$

is a near homeomorphism. According to the Stability Theorem [14, Theorem 15.1], M is homeomorphic to $M \times Q$. Thus, we have the following homeomorphisms:

$$M \cong M \times Q \cong Y \times Q,$$

which completes the proof. $\square$

Proof of Theorem 6.1. (1) By (6.1), $cb_0(\mathbb{R}^n)$ is $O(n)$-homeomorphic to $L_0(n)\times E(n)$. This implies that the orbit spaces $cb_0(\mathbb{R}^n)/K$ and $\frac{L_0(n)\times E(n)}{K}$ are homeomorphic. For this reason, it is enough to prove that $\frac{L_0(n)\times E(n)}{K}$ is a Q-manifold.

Suppose that $\frac{L_0(n)\times E(n)}{K}$ is equipped with the metric D^* induced by D as we have defined in equality (2.1).

By Proposition 5.1, $L(n) \in O(n)$-AR and by Corollary 3.9(2), $E(n) \in O(n)$-AR. Consequently, the product $L_0(n) \times E(n)$ is a locally compact $O(n)$-ANR, which in turn implies that $L_0(n)\times E(n) \in K$-AR (see, e.g., [27]). Then, by Theorem 2.3, the K-orbit space $\frac{L_0(n)\times E(n)}{K}$ is a locally compact ANR.

Let f_ε and h_ε be the maps from Propositions 5.4 and 5.5, respectively. Consider the following maps:

$$f = f_\varepsilon \times Id : L_0(n) \times E(n) \to L_0(n) \times E(n),$$

$$h = h_\varepsilon \times Id : L_0(n) \times E(n) \to L_0(n) \times E(n),$$

where Id denotes the identity map of $E(n)$. Since f_ε and h_ε are $O(n)$-maps with disjoint images, f and h are so. Then they induce continuous maps

$$\widetilde{f},\ \widetilde{h} : \frac{L_0(n)\times E(n)}{K} \to \frac{L_0(n)\times E(n)}{K}$$

which make the followings diagrams commutative:

$$\begin{tikzcd} L_0(n)\times E(n) \arrow[r, "f"] \arrow[d, "p"'] & L_0(n)\times E(n) \arrow[d, "p"] \\ \frac{L_0(n)\times E(n)}{K} \arrow[r, dashed, "\widetilde{f}"'] & \frac{L_0(n)\times E(n)}{K} \end{tikzcd} \qquad \begin{tikzcd} L_0(n)\times E(n) \arrow[r, "h"] \arrow[d, "p"'] & L_0(n)\times E(n) \arrow[d, "p"] \\ \frac{L_0(n)\times E(n)}{K} \arrow[r, dashed, "\widetilde{h}"'] & \frac{L_0(n)\times E(n)}{K} \end{tikzcd}.$$

Since, $d_H\big(f_\varepsilon(A), A\big) < \varepsilon$, we infer that

$$\begin{aligned} D\big(f(A,E),(A,E)\big) &= D\big((f_\varepsilon(A),E),(A,E)\big) \\ &= d_H\big(f_\varepsilon(A),A\big) < \varepsilon \end{aligned}$$

Similarly, we can prove that $D\big(h(A,E),(A,E)\big) < \varepsilon$. Thus, f and h are ε-close to the identity map of $L_0(n) \times E$. Next, using inequality (2.2) we get that $\widetilde{f}$ and $\widetilde{h}$ are ε-close to the identity map of $\frac{L_0(n)\times E(n)}{K}$.

Finally, since $\operatorname{Im}\widetilde{f} = (\operatorname{Im} f)/K$, $\operatorname{Im}\widetilde{h} = (\operatorname{Im} h)/K$ and $\operatorname{Im} f \cap \operatorname{Im} h = \emptyset$, we infer that $\operatorname{Im}\widetilde{f} \cap \operatorname{Im}\widetilde{h} = \emptyset$. Consequently, due to Toruńczyk's Characterization Theorem ([26, Theorem 1]), $\frac{L_0(n)\times E}{K}$ is a Q-manifold, as required.

(2) Since, by Corollary 3.9(2), $cb(\mathbb{R}^n)$ and $L(n)\times E(n)$ are $O(n)$-homeomorphic, the K-orbits spaces $cb(\mathbb{R}^n)/K$ and $\frac{L(n)\times E(n)}{K}$ are homeomorphic. On the other hand, $cb(\mathbb{R}^n)$ is an $O(n)$-AR ([8, Corollary 4.8]), which in turn implies that $cb(\mathbb{R}^n) \in K$-AR (see, e.g., [27]). Then, Theorem 2.3 implies that $cb(\mathbb{R}^n)/K \cong \frac{L(n)\times E(n)}{K}$ is an AR. By the previous case (1), $cb_0(\mathbb{R}^n)/K$ is a Q-manifold while its complement in $cb(\mathbb{R}^n)/K$ is a Z-set (see Corollary 6.3). Now a result of Toruńczyk [26, §3] yields that $cb(\mathbb{R}^n)/K$ is a Q-manifold too.

Furthermore, by Corollary 3.10, $E(n)$ is an $O(n)$-AR, which in turn implies that $E(n) \in K$-AR (see, e.g., [27]). Then, according to Theorem 2.3, the orbit space $E(n)/K$ is an AR.

Since, by Proposition 6.4, the map

$$\widetilde{\pi} : \frac{L(n)\times E(n)}{K} \to E(n)/K$$

is proper and has contractible fibers, it is a CE-map (see [14, Ch. XIII]) between AR's. Since $\frac{cb(\mathbb{R}^n)}{K} \cong \frac{L(n)\times E(n)}{K}$ is a Q-manifold, Edwards' Theorem 6.5 yields that $cb(\mathbb{R}^n)/K$ is homeomorphic to $\big(E(n)/K\big)\times Q$, as required. □

7. Orbit spaces of $cc(\mathbb{R}^n)$

In this section we shall prove the following two main results:

Theorem 7.1. For every closed subgroup $K \subset O(n)$ that acts non-transitively on $\mathbb{S}^{n-1}$, the orbit space $cc(\mathbb{R}^n)/K$ is homeomorphic to the punctured Hilbert cube.

Theorem 7.2. The orbit space $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to the open cone over $\mathrm{BM}(n)$.

The proofs are preceded by some preparation.

Lemma 7.3. The map ν defined in (4.1) is proper and has contractible fibers.

Proof. Clearly, ν is onto. Take a compact subset $C \subset [0,\infty)$. Let b be the supremum of C and denote by N_b the closed ball with the radius b centered at the origin of $\mathbb{R}^n$. Clearly, $\nu^{-1}(C)$ is a closed subset of $cc(N_b)$. According to [21, Theorem 2.2], $cc(N_b)$ is compact, and thus, $\nu^{-1}(C)$ is also compact. This shows that ν is a proper map.

We show that for every point $t \in [0,\infty)$ the inverse image $\nu^{-1}(t)$ is contractible. Consider the homotopy $H : \nu^{-1}(t) \times [0,1] \to \nu^{-1}(t)$ defined by the following formula:

$$H(A,s) = sN_t + (1-s)A, \quad A \in \nu^{-1}(t),\ s \in [0,1]. \tag{7.1}$$

It is easy to see that $H(A,s) \in \nu^{-1}(t)$, and hence, H defines a (strict) homotopy of $\nu^{-1}(t)$ to its point $N_t \in \nu^{-1}(t)$. Thus, $\nu^{-1}(t)$ is contractible, as required. $\square$

Since ν is $O(n)$-invariant, it induces, for every closed subgroup $K \subset O(n)$, a continuous map

$$\widetilde{\nu} : cc(\mathbb{R}^n)/K \to [0,\infty)$$

given by

$$\widetilde{\nu}\big(K(A)\big) = \nu(A), \quad K(A) \in cc(\mathbb{R}^n)/K.$$

Proposition 7.4. $\widetilde{\nu}$ is proper and has contractible fibers.

Proof. Clearly, $\widetilde{\nu}$ is an onto map. Let us denote by $p : cc(\mathbb{R}^n) \to cc(\mathbb{R}^n)/K$ the K-orbit map. Then, we have the following commutative diagram:

$$\begin{tikzcd} cc(\mathbb{R}^n) \arrow[r, "\nu"] \arrow[d, "p"'] & {[0,\infty)} \\ \frac{cc(\mathbb{R}^n)}{K} \arrow[ur, "\widetilde{\nu}"'] & \end{tikzcd}$$

If $C \subset [0,\infty)$ is a compact set, then

$$\widetilde{\nu}^{-1}(C) = \{K(A) \mid \nu(A) \in C\} = p\big(\nu^{-1}(C)\big)$$

which is compact because ν is proper and p is continuous. This yields that $\widetilde{\nu}$ is a proper map.

To finish the proof, let us show that $\widetilde{\nu}^{-1}(t)$ is contractible for every $t \in [0,\infty)$. Consider the homotopy H defined in (7.1). Observe that H is equivariant. Indeed, for every $g \in O(n)$ one has:

$$H(gA,s) = sN_t + (1-s)gA = sgN_t + (1-s)gA = g\big(sN_t + (1-s)A\big) = gH(A,s). \tag{7.2}$$

Hence, H induces a homotopy $\widetilde{H} : \widetilde{\nu}^{-1}(t) \times [0,1] \to \widetilde{\nu}^{-1}(t)$ defined as follows:

$$\widetilde{H}\big(K(A),s\big) = K\big(H(A,s)\big).$$

Clearly, $\widetilde{H}$ is a contraction to the point $K(N_t)$, which proves that $\widetilde{\nu}^{-1}(t)$ is contractible, as required. $\square$

Proposition 7.5. The complement

$$\frac{cc(\mathbb{R}^n)}{K} \setminus \frac{cb(\mathbb{R}^n)}{K}$$

is a Z-set in $cc(\mathbb{R}^n)/K$.

Proof. For every positive ε, the map $\zeta_\varepsilon : cc(\mathbb{R}^n) \to cb(\mathbb{R}^n)$ defined by

$$\zeta_\varepsilon(A) = A_\varepsilon = \{x \in \mathbb{R}^n \mid d(x, A) \leq \varepsilon\}$$

is an $O(n)$-equivariant map which is ε-close to the identity map of $cc(\mathbb{R}^n)$. Hence, for every closed subgroup $K \subset O(n)$ it induces a continuous map

$$\widetilde{\zeta}_\varepsilon : cc(\mathbb{R}^n)/K \to cb(\mathbb{R}^n)/K.$$

Since the Hausdorff metric d_H is $O(n)$-invariant it then follows that d_H induces a metric in $cc(\mathbb{R}^n)/K$ as defined in the equality (2.1). Then, by virtue of inequality (2.2), the map $\widetilde{\zeta}_\varepsilon$ is ε-close to the identity map of $cc(\mathbb{R}^n)/K$. This proves that the set

$$\frac{cc(\mathbb{R}^n) \setminus cb(\mathbb{R}^n)}{K} = \frac{cc(\mathbb{R}^n)}{K} \setminus \frac{cb(\mathbb{R}^n)}{K}$$

is a Z-set in $cc(\mathbb{R}^n)/K$. □

Proof of Theorem 7.1. Since by Theorem 6.1, $cb(\mathbb{R}^n)/K$ is a Q-manifold and the complement $\frac{cc(\mathbb{R}^n)}{K} \setminus \frac{cb(\mathbb{R}^n)}{K}$ is a Z-set, it follows from [26, §3] that $cc(\mathbb{R}^n)/K$ is also a Q-manifold.

Next, since by Proposition 7.4, the map $\widetilde{\nu}\colon cc(\mathbb{R}^n)/K \to [0, \infty)$ is proper and has contractible fibers, it is a CE-map (see [14, Ch. XIII]). Then we can use Edwards' Theorem 6.5 to conclude that $cc(\mathbb{R}^n)/K$ is homeomorphic to $[0, \infty) \times Q$. As shown in the proof of [14, Theorem 12.2], the product $[0, \infty) \times Q$ is homeomorphic to the punctured Hilbert cube, which completes the proof. □

Now we pass to the proof of Theorem 7.2.

The open cone over a topological space X is defined to be the quotient space

$$\mathrm{OC}(X) = X \times [0, \infty)/X \times \{0\}.$$

We will denote by $[A, t]$ the equivalence class of the pair $(A, t) \in X \times [0, \infty)$ in this quotient space. It is evident that $[A, t] = [A', t']$ iff $t = 0 = t'$ or $A = A'$ and $t = t'$. For convenience, the class $[A, 0]$ will be denoted by θ.

Denote the open cone over $M(n)$ by $\widetilde{M}(n)$. The orthogonal group $O(n)$ acts continuously on $\widetilde{M}(n)$ by the following rule:

$$g * [A, t] = [gA, t].$$

Proposition 7.6. The hyperspace $cc(\mathbb{R}^n)$ is $O(n)$-homeomorphic to $\widetilde{M}(n)$.

Proof. Define $\Phi : cc(\mathbb{R}^n) \to \widetilde{M}(n)$ by the formula:

$$\Phi(A) = \begin{cases} \theta, & \text{if } A = \{0\}, \\ [r(A), \nu(A)], & \text{if } A \neq \{0\}, \end{cases}$$

where ν and r are the maps defined in (4.1) and (4.2), respectively.

Since r is $O(n)$-equivariant and ν is $O(n)$-invariant, we infer that Φ is $O(n)$-equivariant.

Clearly, Φ is a bijection with the inverse map $\Phi^{-1} : \widetilde{M}(n) \to cc(\mathbb{R}^n)$ given by

$$\Phi^{-1}([A, t]) = tA.$$

Continuity of the restrictions $\Phi|_{cc(\mathbb{R}^n)\setminus\{0\}}$ and $\Phi^{-1}|_{\widetilde{M}(n)\setminus\{\theta\}}$ is evident. Let us prove the continuity of Φ at $\{0\}$ and the continuity of Φ^{-1} at θ, simultaneously.

Let $\varepsilon > 0$ and let O_ε be the open ε-ball in $cc(\mathbb{R}^n)$ centered at $\{0\}$. Denote $U_\varepsilon = \{[A, t] \in \widetilde{M}(n) \mid t < \varepsilon\}$. Since U_ε is an open neighborhood of θ in $\widetilde{M}(n)$, it is enough to prove that $\Phi(O_\varepsilon) = U_\varepsilon$.

If $B \in O_\varepsilon$ then $B \subset N(\{0\}, \varepsilon)$, and hence, $\nu(B) < \varepsilon$. This proves that $\Phi(B) = [r(B), \nu(B)] \in U_\varepsilon$, implying that

$$\Phi(O_\varepsilon) \subset U_\varepsilon. \tag{7.3}$$

On the other hand, if $[A, t] \in U_\varepsilon$ then $t < \varepsilon$, implying that $tA \subset N(\{0\}, \varepsilon)$. This yields that for every $a \in A$, $d(ta, 0) < \varepsilon$. In particular, $0 \in N(tA, \varepsilon)$, and hence, $d_H(\{0\}, tA) < \varepsilon$. Thus, $\Phi^{-1}(U_\varepsilon) \subset O_\varepsilon$ and

$$U_\varepsilon = \Phi\big(\Phi^{-1}(U_\varepsilon)\big) \subset \Phi(O_\varepsilon). \tag{7.4}$$

Combining (7.3) and (7.4) we get the required equality $\Phi\big(O(\{0\}, \varepsilon)\big) = U_\varepsilon$.

□

Since Φ is an $O(n)$-homeomorphism, it induces a homeomorphism between the $O(n)$-orbit spaces, $cc(\mathbb{R}^n)/O(n)$ and $\widetilde{M}(n)/O(n)$. Thus, we have the following:

Corollary 7.7. The orbit spaces $cc(\mathbb{R}^n)/O(n)$ and $\widetilde{M}(n)/O(n)$ are homeomorphic.

Lemma 7.8. For every closed subgroup $K \subset O(n)$, the orbit space $\widetilde{M}(n)/K$ is homeomorphic to the open cone over $M(n)/K$.

Proof. The map $\Psi : \widetilde{M}(n)/K \to \mathrm{OC}\big(M(n)/K\big)$ defined by the rule:

$$\Psi\big(K[A,t]\big) = [K(A), t],$$

is a homeomorphism. □

Proof of Theorem 7.2. According to Corollary 7.7 and Lemma 7.8, the orbit space $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to the open cone $\mathrm{OC}\big(M(n)/O(n)\big)$. By Corollary 4.16, $M(n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $\mathrm{BM}(n)$, and hence, $cc(\mathbb{R}^n)/O(n)$ is homeomorphic to $\mathrm{OC}\big(\mathrm{BM}(n)\big)$, as required. □

7.1. **Conic structure of $cc(\mathbb{R}^n)$ and related spaces.** It is easy to see that $\mathbb{R}^n$ is $O(n)$-homeomorphic to the open cone over $\mathbb{S}^{n-1}$. This conic structure induces a conic structure in $cc(\mathbb{R}^n)$ as it was shown in Proposition 7.6.

Furthermore, the $O(n)$-homeomorphism between $cc(\mathbb{R}^n)$ and $\widetilde{M}(n)$, in combination with Lemma 7.8, yields the following:

Theorem 7.9. For every closed subgroup $K \subset O(n)$, the K-orbit space $cc(\mathbb{R}^n)/K$ is homeomorphic to the open cone $\mathrm{OC}\big(M(n)/K\big)$

On the other hand, if we restrict the $O(n)$-homeomorphism from Proposition 7.6 to $cc(\mathbb{B}^n)$, we get an $O(n)$-homeomorfism between $cc(\mathbb{B}^n)$ and the cone over $M(n)$.

As in Lemma 7.8, we can prove that the K-orbit space of the cone over $M(n)$ is homeomorphic to the cone over $M(n)/K$ for every closed subgroup K of $O(n)$. This implies the following result:

Proposition 7.10. For every closed subgroup $K \subset O(n)$, the K-orbit space $cc(\mathbb{B}^n)/K$ is homeomorphic to the cone over $M(n)/K$.

Corollary 7.11. For every closed subgroup $K \subset O(n)$ that acts non-transitively on the unit sphere $\mathbb{S}^{n-1}$, the K-orbit space $cc(\mathbb{B}^n)/K$ is homeomorphic to the Hilbert cube.

Proof. By Proposition 7.10, the K-orbit space $cc(\mathbb{B}^n)/K$ is homeomorphic to the cone over $M(n)/K$. Since K acts non-transitively on $\mathbb{S}^{n-1}$, we infer from Corollary 4.13 that $M(n)/K$ is homeomorphic to the Hilbert cube. Thus, $cc(\mathbb{B}^n)/K$ is homeomorphic to the cone over the Hilbert cube, which according to [14, Theorem 12.2], is homeomorphic to the Hilbert cube itself. □

On the other hand, Theorem 4.16 and Proposition 7.10 imply our final result:

Corollary 7.12. The orbit space $cc(\mathbb{B}^n)/O(n)$ is homeomorphic to the cone over the Banach-Mazur compactum $\mathrm{BM}(n)$.

It is well known that the Banach-Mazur compactum $\mathrm{BM}(n)$ is an absolute retract for all $n \geq 2$ (see [5]) and the only compact absolute retract that is homeomorphic to its own cone is the Hilbert cube (see, e.g., [28, Theorem 8.3.2]). Therefore, it follows from Corollary 7.12 and Theorem 4.16 that Pelczyński's question of whether $\mathrm{BM}(n)$ is homeomorphic to the Hilbert cube is equivalent to the following one:

Question 7.13. Are the two orbit spaces $cc(\mathbb{B}^n)/O(n)$ and $M(n)/O(n)$ homeomorphic?

In conclusion we would like to formulate two more questions suggested by the referee of this paper.

Question 7.14. What is the topological type of the pair $\big(cc(\mathbb{R}^n), cb(\mathbb{R}^n)\big)$?

For any $0 \leq k \leq n$, define

$$cc_{\geq k}(\mathbb{R}^n) = \{A \in cc(\mathbb{R}^n) \mid \dim A \geq k\}$$

and observe that $cb(\mathbb{R}^n) = cc_{\geq n}(\mathbb{R}^n)$ and $cc(\mathbb{R}^n) = cc_{\geq 0}(\mathbb{R}^n)$.

Question 7.15. What is the topological structure of the spaces $cc_{\geq k}(\mathbb{R}^n)$ and of the complements $cc_k(\mathbb{R}^n) = cc_{\geq k}(\mathbb{R}^n) \setminus cc_{\geq k+1}(\mathbb{R}^n)$ for $0 \leq k < n$?

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