ORBIT SPACES OF HILBERT MANIFOLDS

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Abstract. Let G be a compact group acting on a Polish group X by means of automorphisms. It is proved that the orbit space X/G is an ℓ_2-manifold (resp., homeomorphic to ℓ_2) provided X is a G-ANR (resp., G-AR) and the fixed point set X^G is not locally compact. It is also proved that if a compact group G acts affinely on a separable closed convex subset K of a Fréchet space with a non-locally compact fixed point set K^G, then the orbit space K/G is homeomorphic to ℓ_2. In particular, (1) if $C(Y, X)$ denotes the space of all maps from a compact metric G-space Y to a non-locally compact Polish ANR (resp., AR) group X, endowed with the compact-open topology and the induced action of G, then the orbit space $C(Y, X)/G$ is an ℓ_2-manifold (resp., homeomorphic to ℓ_2), and (2) if X is an infinite-dimensional separable Fréchet G-space and $cc(X)$ denotes the hyperspace of all non-empty compact convex subsets of X, endowed with the Hausdorff metric topology and the induced action of G, then the orbit space $cc(X)/G$ is homeomorphic to ℓ_2, whenever the fixed point set $cc(X)^G$ is not locally compact.

1. Introduction

All spaces in this paper are assumed to be non-discrete and without isolated points, except for acting groups. As usual, by a *Polish space* we mean a separable completely metrizable topological space. It is known that every Polish group which is an ANR (resp., AR) is either a Lie group or a manifold modelled on the real separable Hilbert space ℓ_2 (resp., homeomorphic either to a Euclidean space $\mathbb{R}^n$ or to the Hilbert space ℓ_2) (see [9, Theorem 3.2] and [8, Corollary 1]).

It is also known that every non-locally compact separable closed convex subset of a Fréchet space is homeomorphic to ℓ_2 (see [8, Theorem 2]).

Also, the hyperspaces of all non-empty compact convex subsets of infinite-dimensional separable Banach spaces, endowed with the Hausdorff metric topology induced by the norm, are known to be homeomorphic to ℓ_2 (see [15, Proposition 1.2]).

In this paper we consider a Polish group X together with an action of a compact group G by means of automorphisms (see formula (2.6) below) and

2010 Mathematics Subject Classification. 52A07, 46A55, 54B20, 54C55.

Key words and phrases. Polish group, infinite-dimensional separable Fréchet space, function space, hyperspace of compact convex sets, affine action, orbit space, ℓ_2-manifold, absolute retract.

Acknowledgements. The first, second and third authors were supported by CONACYT (México) grants 165195, 204028 and 220836, respectively.

we prove that the orbit space X/G is an ℓ_2-manifold (resp., homeomorphic to ℓ_2) provided X/G is an ANR (resp., AR) and the fixed point set X^G is locally path-connected and has no totally bounded neighborhoods (Theorem 3.2).

As a Corollary, the orbit space X/G is an ℓ_2-manifold (resp., homeomorphic to ℓ_2), provided X is a G-ANR (resp., G-AR) and the fixed point set is not locally compact (Corollary 3.4).

We also consider a compact group G acting affinely on a separable closed convex subset K of a Fréchet space (see formula (2.7) below) and prove that the orbit space K/G is homeomorphic to ℓ_2, if the fixed point set K^G is not locally compact (Theorem 4.2).

These results were inspired by those of T. Dobrowolski and H. Toruńczyk [8]. Lemma 3.1 and Theorem 4.2 below are equivariant versions of [8, Lemma 1] and [8, Theorem 2], respectively, which led to the following important corollaries.

Let G be a compact group, Y a compact metric G-space and X a non-locally compact Polish ANR (resp., AR) group. Denote by $C(Y,X)$ the Polish group of all continuous maps from Y to X, endowed with the compact-open topology and the induced action of G (see formula 2.2 below). Then the orbit space $C(Y,X)/G$ is an ℓ_2-manifold (resp., homeomorphic to ℓ_2) (see Corollary 3.7).

Likewise, let a compact group G act linearly on an infinite-dimensional separable Fréchet space X and denote by $cc(X)$ the hyperspace of all non-empty compact convex subsets of X endowed with the Hausdorff metric topology and the induced action of G (see formulas (2.4) and (2.5) below). Then the orbit space $cc(X)/G$ is homeomorphic to ℓ_2, whenever the fixed point set $cc(X)^G$ is not locally compact (Corollary 4.4).

As a by-product, we obtain an alternative proof of Proposition 1.1 below, which is valid for the class of infinite-dimensional separable Fréchet spaces.

Proposition 1.1. [15, Proposition 2.1] *For every infinite-dimensional separable Banach space X, the hyperspace $cc(X)$ is homeomorphic to the Hilbert space ℓ_2.*

2. Preliminaries

We refer the reader to the monographs [7] and [12] for the basic notions of the theory of G-spaces. However, we recall here some special definitions and results that will be used throughout the paper.

All maps between topological spaces are assumed to be continuous. A map $f : X \to Y$ between G-spaces is called *G-equivariant* (or simply *equivariant*) if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. In case G acts trivially on Y (i.e., $gy = y$ for every $g \in G$ and $y \in Y$), an equivariant map $f : X \to Y$ is called *invariant*.

Let (X, d) be a metric G-space. If $d(gx, gy) = d(x, y)$ for every $x, y \in X$ and $g \in G$, then we say that d is a *G-invariant* metric. That is, every $g \in G$ acts, in fact, as an isometry of X with respect to the metric d.

Proposition 2.1. [2, Proposition 5] *Let G be a compact group and (X, d) a metric G-space. Then the formula:*

$$\hat{d}(x, y) = \sup_{g \in G} d(gx, gy), \qquad x, y \in X$$

defines a compatible G-invariant metric on X. Moreover,

(1) *If d is complete, then $\hat{d}$ is complete*
(2) *If X is a topological group and d is right or left-invariant, then $\hat{d}$ is right or left-invariant, respectively.*

Let G be a compact group and X a metric G-space with a G-invariant metric d. It is well-known that the quotient topology of the orbit space X/G is generated by the metric

$$d^*(G(x), G(y)) = \inf_{g \in G} d(x, gy), \quad G(x), G(y) \in X/G$$

(see, e.g., [12, Proposition 1.1.12]). Evidently,

$$d^*(G(x), G(y)) \leq d(x, y), \quad x, y \in X. \tag{2.1}$$

For a given topological group G, a metrizable G-space X is called a *G-equivariant absolute neighborhood retract* (denoted by $X \in G$-ANR) if for any metrizable G-space Z containing X as an invariant closed subset, there exist an invariant neighborhood U of X in Z and an equivariant retraction $r : U \to X$. If we can always take $U = Z$, then we say that X is a *G-equivariant absolute retract* (denoted by $X \in G$-AR).

Theorem 2.2 ([3, Theorem 8])**.** *Let G be a compact group and X a G-*ANR *(resp., G-*AR*). Then the orbit space X/G is an* ANR *(resp.,* AR*).*

A point x_0 in a G-space X is called a *G-fixed point* if $gx_0 = x_0$ for every $g \in G$. The set of all G-fixed points is denoted by X^G.

Theorem 2.3 ([1, Theorem 7])**.** *Let G be a compact group and X a G-*ANR *(resp., G-*AR*). Then the fixed point set X^G is an* ANR *(resp.,* AR*).*

By a *linear space* we mean a real topological vector space. A metric d for a linear space X is called *invariant*, if d is compatible with the topology of X and $d(x + z, y + z) = d(x, y)$ for every $x, y, z \in X$.

A *Fréchet space* is a locally convex complete metric linear space with an invariant metric (see [6, Chapter I, § 6]).

Let G be a topological group and X a linear space. We call X a *linear G-space* if it is a G-space endowed with a linear action of G, i.e., if

$$g(\alpha x + \beta y) = \alpha(gx) + \beta(gy)$$

for every $g \in G$, $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. If, in addition, X is a Fréchet space endowed with a complete metric which simultaneously is invariant and G-invariant, then we call X a *Fréchet G-space.*

In particular, if a compact group G acts linearly on a Fréchet space X with complete invariant metric d, then Proposition 2.1 implies that

$$\hat{d}(x, y) = \sup_{g \in G} d(gx, gy), \quad x, y \in X$$

is an invariant and G-invariant complete metric on X.

Theorem 2.4 ([1, Theorem 2])**.** *Let G be a compact group acting linearly on a locally convex metric linear space X and let K be an invariant complete convex subset of X. Then K is a G-AR.*

For a compact group G, a compact G-space Y and a space X, we denote by $C(Y,X)$ the space of all maps from Y to X endowed with the compact-open topology and the induced action $G \times C(Y,X) \to C(Y,X)$:

$$(gf)(y) = f(g^{-1}y), \quad g \in G, \quad y \in Y, \quad f \in C(Y,X) \tag{2.2}$$

(see [1, Lemma 1]).

If X is a topological group with the identity element denoted by 1, then $C(Y,X)$ becomes a topological group with pointwise defined operations, i.e.,

$$(f \cdot h)(y) = f(y) \cdot g(y), \quad f,h \in C(Y,X), \quad y \in Y$$

and

$$f^{-1}(y) = \big(f(y)\big)^{-1}, \quad y \in Y.$$

The identity element is just the constant map 1 (see e.g., [5, § 3]).

Furthermore, if X admits a complete metric d, then the supremum metric on $C(Y,X)$

$$\rho(f,j) = \sup_{y \in Y} d\big(f(y), j(y)\big), \quad f,j \in C(Y,X)$$

is also complete, and by Proposition 2.1, the metric

$$\hat{\rho}(f,j) = \sup_{g \in G} \rho(gf, gj), \quad f,j \in C(Y,X) \tag{2.3}$$

defines a complete and G-invariant metric on $C(Y,X)$. If, in addition, Y is metrizable and X is separable, then $C(Y,X)$ is separable (see [10, Theorem 3.4.16]). Note that due to compactness of Y, the topology induced by the metric ρ, and hence, the one induced by the metric $\hat{\rho}$ on $C(Y,X)$, is just the compact-open one. Note also that if X is a linear space and $Y = G$ is endowed with the following action of G

$$(g,y) \mapsto yg^{-1}, \quad g,y \in G$$

then $C(G,X)$ is a linear space and the action (2.2) is linear and becomes:

$$(gf)(y) = f(yg), \quad g,y \in G, \quad f \in C(Y,X).$$

The following Theorem belongs to Y. Smirnov (see [17, Theorem 2] and [1, Theorem 2]).

Theorem 2.5. *Let G be a compact group, X a Tychonoff G-space and $h : X \to Y$ a closed embedding of X into a locally convex linear space Y. Then the map $\tilde{h} : X \to C(G,Y)$ defined by the rule*

$$\tilde{h}(x)(g) = h(gx), \qquad x \in X, \qquad g \in G$$

is a closed equivariant embedding of X into the locally convex linear G-space $C(G,Y)$.

Theorem 2.6 ([2, Theorem 8])**.** *Let G be a compact group, Y a compact G-space and X an* ANR *(resp.,* AR*). Then $C(Y,X)$ is a G-*ANR *(resp., G-*AR*).*

Let (X,d) be a metric linear G-space. By $cc(X)$ we denote the hyperspace of all non-empty compact convex subsets of X endowed with the Hausdorff metric:

$$d_H(A,B)=\max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\},\quad A,B\in cc(X) \tag{2.4}$$

and the induced action of G:

$$(g,A)\mapsto gA=\{ga \mid a\in A\},\quad g\in G,\quad A\in cc(X). \tag{2.5}$$

The following theorem is an extension of H. Radström's embedding theorem [13, Theorem 2] and is due to K. Schmidt (see [16, § 5, 6 and 7]). Recall that a *monoid* is a set together with an associative operation and identity element.

Theorem 2.7. *Let X be an infinite-dimensional separable Fréchet space X. Then the hyperspace $cc(X)$ embeds as a convex Polish submonoid of an infinite-dimensional separable Fréchet space.*

We say that a topological group G acts on a monoid $(X,\cdot)$ *by means of automorphisms* if

$$g(x\cdot y)=gx\cdot gy \tag{2.6}$$

for every $g\in G$ and $x,y\in X$, i.e., every $g\in G$ is an automorphism of X.

Analogously, we say that a topological group G acts *affinely* on a convex subset K of a linear space if

$$g\Big(\sum_{i=1}^{n} t_i x_i\Big)=\sum_{i=1}^{n} t_i g x_i \tag{2.7}$$

whenever $x_i\in X$, $t_i\in[0,1]$ and $\sum_{i=1}^{n} t_i=1$, i.e., every $g\in G$ is a self-affine-homeomorphism of K.

A *separable Hilbert manifold* or an *ℓ_2-manifold* is a separable completely metrizable space that admits an open cover each member of which is homeomorphic to an open subset of the Hilbert space ℓ_2. We refer the reader to [18] and [19] (see also [20]) for the theory of ℓ_2-manifolds. Nevertheless, below we recall the characterization Theorem for ℓ_2-manifolds due to H. Toruńczyk as well as a result of J. Mogilski [11], which will be important in the proof of Theorem 4.2.

Throughout the rest of the paper we let D denote the countable disjoint union of n-cells $\mathbb{I}^n := [-1,1]^n$, $n\geq 0$, i.e.,

$$D=\bigsqcup_{n\geq 0}\mathbb{I}^n.$$

Theorem 2.8 ([19, Corollary 3.2] and [8, § 2 Condition (*)])**.** *A separable completely metrizable* ANR *(resp.,* AR*) X is an ℓ_2-manifold (resp., homeomorphic to ℓ_2) if and only if there is a compatible metric d on X such that* *given maps $f : D \to X$ and $\alpha : X \to (0,1)$, there is a map $g : D \to X$ with $d\big(g(t), f(t)\big) < \alpha\big(f(t)\big)$ for every $t \in D$ and $\big\{g(\mathbb{I}^n)\big\}_{n\geq 0}$ is a discrete family in X.*

Theorem 2.9 ([11, Corollary 1]). *If the product $X \times Y$ is an ℓ_2-manifold and Y is locally compact, then X is an ℓ_2-manifold.*

3. Orbit spaces of Polish groups

The results of this section are valid for invariant submonoids of Polish groups. For the first lemma, we consider the following situation.

Let H be a topological group with a right-invariant metric ρ and let X be a submonoid of H which is complete with respect to ρ. Further, let a compact group G act on X by means of automorphisms and let d be defined by the rule:

$$d(x,y) = \sup_{g\in G} \rho(gx, gy), \qquad x, y \in X.$$

Then d is a compatible right-invariant and G-invariant complete metric on X (see Proposition 2.1).

With the above notation, the following lemma is a modification of [8, Lemma 1].

Lemma 3.1. *Let G be a compact group acting by means of automorphisms on a submonoid X of a topological group H and let d be a compatible right-invariant and G-invariant complete metric on X. If the fixed point set X^G is locally path connected at the identity $1 \in X$ and no neighborhood of 1 in X^G is totally bounded in the metric d, then given maps $f : D \to X/G$ and $\alpha : X/G \to (0,1)$ there is a map $\widetilde{g} : D \to X/G$ such that $d^*\big(\widetilde{g}(t), f(t)\big) < \alpha\big(f(t)\big)$ for every $t \in D$ and $\{\widetilde{g}(\mathbb{I}^n)\}_{n\geq 0}$ is discrete in X/G.*

Proof. Let $\pi : X \to X/G$ be the orbit map and consider the pull-back

$$C = \big\{(t,x) \in D \times X \mid f(t) = \pi(x)\big\}$$

of X via f with the diagonal action of G. Identify D with the orbit space C/G and write $\phi : C \to D$ for the orbit map and $p : C \to X$ for the equivariant projection to X (see [7, Chapter 1,§ 6(B)]). We have the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{p} & X \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{f} & X/G \end{array}$$

For every $k \geq 1$, let

$$D_k = \big\{\phi(t,x) \in D \mid (\alpha\circ f)\big(\phi(t,x)\big) \geq 1/k\big\}, \quad C_k = \phi^{-1}(D_k) \quad \text{and} \quad \mathbb{L}^{k-1} = \phi^{-1}(\mathbb{I}^{k-1}).$$

Assume without loss of generality that $D_2 = \emptyset$. We construct a sequence of equivariant maps $\{g_k : C \to X\}_{k\geq 1}$ and a sequence of positive numbers $\{\epsilon_k\}_{k\geq 1}$ such that for every $k \geq 1$ the following conditions are satisfied:

$(1)_k$ $g_k = p$ in $C\backslash C_{k+1}$ and $g_k = g_{k-1}$ in C_{k-1},

$(2)_k$ $\operatorname{dist}\big(g_k(\mathbb{L}^n \cap C_k), g_k(\mathbb{L}^m)\big) > \epsilon_k, \quad m < n,$

$(3)_k$ $d\big(g_k(t,x), g_{k-1}(t,x)\big) < \frac{1}{4}\epsilon_{k-1}, \quad (t,x) \in C,$

$(4)_k$ $\epsilon_k < \min\{\frac{1}{k}, \frac{\epsilon_{k-1}}{4}\}.$

Indeed, define $g_0 = f$ and $\epsilon_0 = 1$ and suppose that g_{k-1} and ϵ_{k-1} are known. By $(2)_{k-1}$, compactness of G and the fact that C_{k-1} is an invariant subset of C, we can find an invariant neighborhood U of C_{k-1} in C such that

$(5)_k$ $\operatorname{dist}\big(g_{k-1}(\mathbb{L}^n \cap U), g_{k-1}(\mathbb{L}^m)\big) > \frac{3\epsilon_{k-1}}{4}, \quad m < n.$

Let B be a path connected neighborhood of 1 in X^G with $\operatorname{diam} B < \frac{1}{4}\epsilon_{k-1}$. Since B is not totally bounded, there exists $\epsilon_k > 0$ satisfying $(4)_k$ and no compact set in H is an ϵ_k-net for B. Define $g_k(\mathbb{L}^0) = f(\mathbb{L}^0)$ and assume that $g_k(\mathbb{L}^0 \sqcup \cdots \sqcup \mathbb{L}^{n-1})$ is already known. Let

$$Z = \big\{ae^{-1} \in H \mid a, e \in g_k(\mathbb{L}^0 \sqcup \cdots \sqcup \mathbb{L}^{n-1}) \cup g_{k-1}(\mathbb{L}^n)\big\}.$$

Due to the choice of ϵ_k and the fact that Z is a compact set in H, there is a point $b \in B$ such that

$$\operatorname{dist}(b, Z) > \epsilon_k. \tag{3.1}$$

Let $h : I \to B$ be a path such that $h(0) = 1$ and $h(1) = b$, and let $\omega : C \to I$ be a Urysohn map such that

$$\omega(C_k \backslash U) \subset \{1\} \quad \text{and} \quad \omega\big((C \backslash C_{k+1}) \sqcup C_{k-1}\big) \subset \{0\}.$$

Then the map $\upsilon : C \to I$ defined by

$$\upsilon(d, x) = \sup_{g \in G} \omega(d, gx), \quad (d, x) \in C,$$

is invariant and also satisfies

$$\upsilon(C_k \backslash U) \subset \{1\} \quad \text{and} \quad \upsilon\big((C \backslash C_{k+1}) \sqcup C_{k-1}\big) \subset \{0\}.$$

Define $g_k|_{\mathbb{L}^n} : \mathbb{L}^n \to X$ by the rule

$$g_k(t, x) = h\big(\upsilon(t, x)\big) \cdot g_{k-1}(t, x), \quad (t, x) \in \mathbb{L}^n.$$

Since $B \subset X^G$, the map $g_k|_{\mathbb{L}^n}$ is equivariant. Indeed, let $q \in G$ and $(t, x) \in \mathbb{L}^n$. Then

$$\begin{aligned} g_k(t, qx) &= h\big(\upsilon(t, qx)\big) \cdot g_{k-1}(t, qx) = h\big(\upsilon(t, x)\big) \cdot q g_{k-1}(t, x) \\ &= q\big(h\big(\upsilon(t, x)\big) \cdot g_{k-1}(t, x)\big) = q g_k(t, x). \end{aligned}$$

Condition $(3)_k$ for $g_k|_{\mathbb{L}^n}$ follows from the invariance of the metric d and the choice of B. Condition $(2)_k$ is also satisfied. Indeed, let $(t, x) \in \mathbb{L}^n \cap C_k$, $(s, y) \in \mathbb{L}^m$ and $m < n$. If $(t, x) \notin U$, then, using inequality (3.1) we get

$$g_k(t, x) = b \cdot g_{k-1}(t, x) \quad \text{and} \quad d\big(g_k(t, x), g_k(s, y)\big) = d\Big(b, g_k(s, y)\big(g_{k-1}(t, x)\big)^{-1}\Big) > \epsilon_k.$$

If $(t, x) \in U$, then $(5)_k$ implies that

$$\frac{3\epsilon_{k-1}}{4} < d\big(g_{k-1}(t, x), g_{k-1}(s, y)\big).$$

Using the triangle inequality, we get

$$\frac{3\epsilon_{k-1}}{4} < d\big(g_{k-1}(t,x), g_k(t,x)\big) + d\big(g_k(t,x), g_k(s,y)\big) + d\big(g_k(s,y), g_{k-1}(s,y)\big).$$

Now, condition $(3)_k$ for $g_k\,|_{\mathbb{L}^n}$ implies

$$\frac{\epsilon_{k-1}}{4} < d\big(g_k(t,x), g_k(s,y)\big).$$

Hence, by $(4)_k$,

$$\epsilon_k < \frac{\epsilon_{k-1}}{4} \leq \operatorname{dist}\big(g_k(\mathbb{L}^n \cap C_k), g_k(\mathbb{L}^m)\big).$$

Next, if $(t,x) \in \mathbb{L}^n \cap (C \setminus C_{k+1})$, then $\upsilon(t,x) = 0$, and consequently, by $(1)_{k-1}$,

$$g_k(t,x) = g_{k-1}(t,x) = p(t,x).$$

If $(t,x) \in \mathbb{L}^n \cap C_{k-1}$, then also $\upsilon(t,x) = 0$ and clearly $g_k(t,x) = g_{k-1}(t,x)$. Thus $(1)_k$ holds for $g_k\,|_{\mathbb{L}^n}$ and by induction on n we obtain an equivariant map $g_k : C \to X$ fulfilling conditions $(1)_k - (4)_k$.

By $(1)_k$ and $(3)_k$, $k \geq 1$, there is a well-defined equivariant map $g = \lim g_k$, satisfying

$$d\big(g(t,x), g_k(t,x)\big) \leq \sum_{l \geq k} d\big(g_{l+1}(t,x), g_l(t,x)\big) < \frac{1}{4}\sum_{l \geq k} \epsilon_l < \frac{\epsilon_k}{3}. \tag{3.2}$$

For $(t,x) \in C$, say $(t,x) \in C_k \setminus C_{k-1}$, we have

$$d\big(g(t,x), p(t,x)\big) = d\big(g_k(t,x), g_{k-1}(t,x)\big) \leq \frac{\epsilon_{k-1}}{3} \leq (k+1)^{-1} \leq (\alpha \circ f)\big(\phi(t,x)\big). \tag{3.3}$$

Now we consider the induced maps $\widetilde{g}$, $\widetilde{g}_k : D \to X/G$, $k \geq 1$, of g and g_k, respectively, which are defined by the rules:

$$\widetilde{g}\big(\phi(t,x)\big) = \pi\big(g(t,x)\big) \quad \text{and} \quad \widetilde{g}_k\big(\phi(t,x)\big) = \pi\big(g_k(t,x)\big), \quad \phi(t,x) \in D.$$

Since $g = \lim g_k$ and π is continuous, we also have $\widetilde{g} = \lim \widetilde{g}_k$. By (2.1), (3.2) and (3.3),

$$d^*\Big(\widetilde{g}\big(\phi(t,x)\big), \widetilde{g}_k\big(\phi(t,x)\big)\Big) < \frac{\epsilon_k}{3} \tag{3.4}$$

and for $\phi(t,x) \in D$, say $\phi(t,x) \in D_k \setminus D_{k-1}$, one has

$$d^*\Big(\widetilde{g}\big(\phi(t,x)\big), f\big(\phi(t,x)\big)\Big) \leq (\alpha \circ f)\big(\phi(t,x)\big). \tag{3.5}$$

Now, for every $k \geq 1$, condition $(2)_k$ together with the equivariance of g_k imply the following condition:

$(2')_k$ $\operatorname{dist}\big(\widetilde{g}_k(\mathbb{I}^n \cap D_k), \widetilde{g}_k(\mathbb{I}^m)\big) > \epsilon_k, \quad m < n.$

Indeed, let $\phi(t,x) \in \mathbb{I}^n \cap D_k$, $\phi(s,y) \in \mathbb{I}^m$ and $m < n$. Then $(t,x) \in \mathbb{L}^n \cap C_k$ and $(s,y) \in \mathbb{L}^m$. Hence,

$$\begin{aligned} d^*\Big(\widetilde{g}_k\big(\phi(t,x)\big), \widetilde{g}_k\big(\phi(s,y)\big)\Big) &= \inf_{q \in G} d\big(g_k(t,x), q g_k(s,y)\big) \\ &= \inf_{q \in G} d\big(g_k(t,x), g_k(s,qy)\big) \\ &\geq \operatorname{dist}\big(g_k(\mathbb{L}^n \cap C_k), g_k(\mathbb{L}^m)\big) > \epsilon_k. \end{aligned}$$

Finally, we show that the family $\{\widetilde{g}(\mathbb{I}^n)\}_{n\geq 0}$ is discrete. Assume, by contradiction, that a sequence

$$\widetilde{g}(a_i) \rightsquigarrow \widetilde{x} \in X/G, \qquad a_i \in D \cong C/G$$

and distinct orbits a_i belong to distinct n-cells in D. Then

$$\inf_{i\in\mathbb{N}} (\alpha \circ f)(a_i) > 0$$

otherwise, by inequality (3.5), $\big(f(a_i)\big)$ contains a subsequence $\big(f(a_j)\big)$ such that

$$f(a_j) \rightsquigarrow \widetilde{x}$$

with $(\alpha \circ f)(a_j) \rightsquigarrow 0$, contradicting the fact that $\alpha(\widetilde{x}) > 0$. Therefore, for such a sequence, there is a $k \geq 1$ with $a_i \in D_k$ for every $i \geq 1$ and, by condition $(2')_k$ and inequality (3.4), we get for every $i < j$,

$$d^*\big(\widetilde{g}(a_i), \widetilde{g}(a_j)\big) = d^*\big(\widetilde{g}_k(a_i), \widetilde{g}_k(a_j)\big) > \epsilon_k,$$

contradicting the convergence of $\widetilde{g}(a_i)$. □

Theorem 3.2. *Let a compact group G act on a complete submonoid X of a Polish group by means of automorphisms. Then the orbit space X/G is an ℓ_2-manifold* (*resp., homeomorphic to ℓ_2*)*, if X/G is an* ANR (*resp.,* AR) *and X^G is locally path connected and has no totally bounded neighborhood.*

Proof. It follows directly from Lemma 3.1 and Theorem 2.8. □

Immediate Corollaries are the following:

Corollary 3.3. *Let a compact group G act on a complete submonoid X of a Polish group H by means of automorphisms. Then X/G is an ℓ_2-manifold* (*resp., homeomorphic to ℓ_2*)*, if X/G is an* ANR (*resp.,* AR) *and X^G is a non-locally compact* ANR*.*

Proof. Clearly, X^G is itself a Polish submonoid of H. It then follows from [8, Theorem 1] that X^G is an ℓ_2-manifold. Now the Corollary follows from Theorem 3.2. □

Corollary 3.4. *Let a compact group G act on a complete submonoid X of a Polish group by means of automorphisms. Assume further, that X is a* G-ANR (*resp.,* G-AR)*. Then X/G is an ℓ_2-manifold* (*resp., homeomorphic to ℓ_2*)*, if X^G is not locally compact.*

Proof. By Theorems 2.2 and 2.3, the orbit space X/G and the fixed point set X^G are ANR's (resp., AR's). In any case X^G is a non-locally compact ANR and the Corollary follows from Corollary 3.3. □

In particular, an infinite-dimensional separable Fréchet G-space X is a Polish group. Hence we have the following Corollaries.

Corollary 3.5. *Let a compact group G act linearly on a separable Fréchet space X. Then X/G is homeomorphic to ℓ_2, if X^G is not locally compact.*

Proof. By Theorem 2.4, X is a G-AR. Since X^G is not locally compact, the Corollary follows from Corollary 3.4. □

Remark 3.6. *The non-local compactness assumption in Corollary* 3.5 *is essential. The cyclic group* $\mathbb{Z}_2$ *acts linearly on the Hilbert space* ℓ_2 *by reflection at the origin* $0 \in \ell_2$ *with fixed point set* $\ell_2^{\mathbb{Z}_2} = \{0\}$*. Since the orbit map* $\ell_2 \backslash \{0\} \to \big(\ell_2 \backslash \{0\}\big)/\mathbb{Z}_2$ *is a two fold covering map, the orbit space* $\big(\ell_2 \backslash \{0\}\big)/\mathbb{Z}_2 = (\ell_2/\mathbb{Z}_2) \backslash \big\{\mathbb{Z}_2(0)\big\}$ *is not contractible. Since the complement of each point in* ℓ_2 *is contractible, in fact,* $\ell_2 \backslash \{x\}$ *is homeomorphic to* ℓ_2 *for every* $x \in \ell_2$ *(see e.g.,* [6, Chapter III, § 5, Corollary 5.1]*), the orbit space* $\ell_2/\mathbb{Z}_2$ *is not homeomorphic to* ℓ_2*.*

Corollary 3.7. *Let* G *be a compact group,* Y *a compact metric* G*-space and* X *a Polish* ANR *(resp.,* AR*) group. Then the orbit space* $C(Y, X)/G$ *is an* ℓ_2*-manifold (resp., homeomorphic to* ℓ_2*), if* $C(Y, X)^G$ *is not locally compact (e.g., if* X *is infinite-dimensional).*

Proof. Note that the action (2.2) of G on $C(Y, X)$ is by means of automorphisms. Indeed, let $g \in G$, $f, h \in C(Y, X)$ and $y \in Y$. Then

$$\big(g(f{\cdot}h)\big)(y) = (f{\cdot}h)\big(g^{-1}y\big) = f(g^{-1}y){\cdot}h(g^{-1}y) = (gf)(y){\cdot}(gh)(y) = (gf{\cdot}gh)(y).$$

Thus,

$$g(f \cdot h) = gf \cdot gh.$$

Now Corollary 3.7 follows from Theorem 2.6 and Corollary 3.4.
In the particular case when X is infinite-dimensional, we note that constant maps conform a topological copy of X and they belong to the fixed point set $C(Y, X)^G$, which, by Theorems 2.6 and 2.3, is an AR. Hence, $C(Y, X)^G$, being an infinite-dimensional Polish AR group, is homeomorphic to ℓ_2, by [8, Corollary 1] and Theorem 2.8, and thus, it is not locally compact. ☐

4. Orbit spaces of separable closed convex subsets of Fréchet spaces

We begin this section with the following equivariant embedding result.

Lemma 4.1. *Let a compact group* G *act affinely on a closed convex subset* K *of a locally convex linear space* X*. Then there is a closed affine equivariant embedding of* K *into the locally convex linear* G*-space* $C(G, X)$*.*

Proof. Let $j : K \to C(G, X)$ be given by the rule:

$$j(k)(g) = gk, \quad k \in K, \quad g \in G$$

By Theorem 2.5, j is a closed equivariant embedding. Since G acts affinely on K, j is also an affine map. Indeed, let $n \in \mathbb{N}$, $k_i \in K$ and $t_i \geq 0$ such that $\sum_{i=1}^{n} t_i = 1$. Then for every $g \in G$ we have

$$\begin{aligned} j\Big(\sum_{i=1}^{n} t_i k_i\Big)(g) &= g\sum_{i=1}^{n} t_i k_i = \sum_{i=1}^{n} t_i g k_i = \sum_{i=1}^{n} t_i\Big(j(k_i)(g)\Big) \\ &= \sum_{i=1}^{n}\Big(t_i j(k_i)\Big)(g) = \Big(\sum_{i=1}^{n} t_i j(k_i)\Big)(g). \end{aligned}$$

Hence,

$$j\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i j(x_i),$$

showing that j is an affine map. Thus, K embedds as an invariant closed convex subset into the locally convex linear G-space $C(G, X)$. □

The following theorem is a modification of [8, Theorem 2].

Theorem 4.2. *Let G be a compact group acting affinely on a separable closed convex subset K of a Fréchet space. If K^G is not locally compact, then K/G is homeomorphic to ℓ_2.*

Proof. By Lemma 4.1, we may assume that K is an invariant separable closed convex subset of a Fréchet G-space X. Further, assume without loss of generality that K^G contains the origin 0 of X. Let G act on $X \times \mathbb{R}$ by the rule:

$$\big(g, (x, t)\big) \mapsto (gx, t), \quad g \in G, \quad (x, t) \in X \times \mathbb{R}$$

and let

$$Y = \big\{(x, t) \in X \times [0, \infty) \mid x \in tK\big\}.$$

Clearly, G acts linearly on $X \times \mathbb{R}$ and Y is an invariant closed convex submonoid of $X \times \mathbb{R}$. By Theorem 2.4, Y is a G-AR. Let $Y_0 := Y \backslash \{(0, 0)\}$ and define a map $h : K \times (0, 1) \to Y_0$ by the rule:

$$(x, t) \mapsto (tx, t), \quad (x, t) \in K \times (0, 1).$$

Since G acts linearly on K, the map h is an equivariant homeomorphism. Hence, the induced map

$$\widetilde{h} : \big(K \times (0, 1)\big)/G \to Y_0/G$$

is a homeomorphism. The fact that G acts trivially on $(0, 1)$ and that the orbit of $(0, 0) \in Y$ is just the singleton $\{(0, 0)\}$, imply that the orbit spaces $\big(K \times (0, 1)\big)/G$ and Y_0/G are homeomoprhic to

$$(K/G) \times (0, 1) \quad \text{and} \quad (Y/G)_0 := (Y/G)\backslash\big\{\{(0, 0)\}\big\}$$

respectively. Since $K^G \times \{1\} \subset Y^G = \big(X^G \times [0, \infty)\big) \cap Y$ and K^G is not locally compact, the fixed point set Y^G is neither locally compact. It follows from Corollary 3.4 that the orbit space Y/G is homeomorphic to ℓ_2 and, since points can be deleted from ℓ_2 (see e.g., [6, Chapter III, § 5, Corollary 5.1]), the space $(Y/G)_0$ is homeomorphic to ℓ_2. Consequently, the space $(K/G) \times (0, 1)$ is also homeomorphic to ℓ_2. By Theorems 2.4 and 2.2, the orbit space K/G is an AR. Finally, by Theorem 2.9, the orbit space K/G is homeomorphic to ℓ_2. This completes the proof. □

Proposition 4.3. *Let G be a compact group and X an infinite-dimensional separable Fréchet G-space. Then the hyperspace $cc(X)$ embeds equivariantly as an invariant separable closed convex subset of a Fréchet G-space.*

Proof. By Theorem 2.7, we may assume that $cc(X)$ is a Polish convex subset of a infinite-dimensional separable Fréchet space. Then G acts affinely on $cc(X)$ and by Lemma 4.1, $cc(X)$ embeds equivariantly as an invariant separable closed convex subset of a Fréchet G-space, as required. □

Corollary 4.4. *Let G be a compact group and X an infinite-dimensional separable Fréchet G-space. Then the orbit space $cc(X)/G$ is homeomorphic to ℓ_2, if $cc(X)^G$ is not locally compact.*

Proof. Since fixed points are preserved by equivariant maps, the Corollary follows directly from Proposition 4.3 and Theorem 4.2. □

In contrast with Remark 3.6, we end this paper with the following one.

Remark 4.5. *The hyperspace $cc(\ell_2)$, which is homeomorphic to the Hilbert space ℓ_2, becomes a $\mathbb{Z}_2$-space with the induced action of $\mathbb{Z}_2$ described in Remark 3.6. But in this case, the fixed point set $cc(\ell_2)^{\mathbb{Z}_2}$ is not locally compact. Indeed, for every neighborhood U of $\{0\}$ in $cc(\ell_2)^{\mathbb{Z}_2}$, there is an $\epsilon > 0$, such that for every $n \geq 1$, the segment*

$$A_n := \left\{ta_n + (1-t)(-a_n) \in \ell_2 \;\middle|\; a_{n_n} = \epsilon \text{ and } a_{n_i} = 0, \text{ if } i \neq n, \ t \in [0,1]\right\}$$

belongs to U. Since $d_H(A_n, A_m) = \epsilon$, if $n \neq m$, the sequence $(A_n)_{n\geq 1}$ has no convergent subsequence. Here, the Hausdorff metric d_H is the one induced by the standard metric of ℓ_2. Thus, we conclude that the orbit space $cc(\ell_2)/\mathbb{Z}_2$ is homeomorphic to the Hilbert space ℓ_2.

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