G-Equivariant Selections and Near Selections

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Abstract

Let G a locally compact topological group. Let L be a linear G-space and $Y \subset L$ a metrizable convex proper subset. Let X be a paracompact proper G-space with paracompact orbit space. We will give conditions for Y in order that every equivariant l.s.c. multivalued mapping $\phi : X \Rightarrow Y$ with complete and convex values admits an equivariant selection.

1 Introduction

The classical Michael selection theorem [9] states that every lower semicontinuous multivalued mapping from a paracompact space into the non empty closed and convex sets of a Banach space admits a selection. By following the same method used in Michael [9], it was proved in [12, Theorem 1.4.9] that every lower semicontinuous multivalued mapping from a paracompact space into the non empty complete and convex sets of a normed linear space admits a selection. The proof of these theorems consists on finding an ε-near selection for every positive ε. Then the required selection apears as the limit of a carefully constructed sequence of 2^{-n}-near selections.

In [5] an equivariant generalization of Michael's theorem was proved: If G is a compact group, X is a paracompact G-space and Y is a Banach G-space, then every lower semicontinuous multivalued equivariant map from X into the non empty convex and closed subsets of Y admits an equivariant selection. In the proof of this result, the authors used the vector valued integral with respect to the Haar measure for integrating a (non equivariant) selection in order to obtain the desired equivariant selection. Because the Haar integral was used, the completeness of the codomain Y and the compactness of the group G are necessary.

In the present paper, we will give an equivariant version of Michael's theorem which also generalizes the result in [5] (Corollary 5.5). The idea is to follow the Michael's proof: first we obtain an equivariant near selection (theorems 4.3 and 4.5) and then we use the same method used in [12, Theorem 1.4.9] to obtain an equivariant selection (proposition 5.2 and corollaries 5.3, 5.4, 5.5 and 5.6).

2 Preliminaries

If G is a topological group and X is a G-space, for any $x \in X$ we denote the stabilizer subgroup of x by $G_x = \{g \in G \mid gx = x\}$. For a subset $S \subset X$ and a subgroup $H \subset G$, $H(S)$ denotes the H-saturation of S, i.e., $H(S) = \{hs \mid h \in H, s \in S\}$. If $H(S) = S$ then we say that S is an H-invariant set. In particular, $G(x)$ denotes the G-orbit of x, so that $G(x) = \{gx \in X \mid g \in G\}$. The orbit space is denoted by X/G. For any subgroup $H \subset G$, we will denote by G/H the G-space of cosets $\{gH \mid g \in G\}$ equipped with the action induced by left translations.

A G-space X is called *proper* (in the sense of Palais), if every point $x \in X$ has a neighborhood U such that for any other point $y \in X$ there exists a neighborhood V of y such that $\{g \in G \mid gU \cap V \neq \emptyset\}$ has compact closure in G. Each orbit in a proper G-space is closed, and each stabilizer is compact ([11, Proposition 1.1.4]).

A map $f : X \to Y$ between two G-spaces is called *equivariant* or a *G-map* if $f(gx) = g(fx)$ for every $x \in X$ and $g \in G$.

Let G be a topological group and X a G-space. A G-space Y is called an *equivariant absolute neighborhood extensor* for X (denoted by $Y \in G\text{-ANE}(X)$) if, for any closed invariant subset $A \subset X$ and any equivariant map $f : A \to Y$, there exists an invariant neighborhood U of A in X and an equivariant map $F : U \to Y$ such that $F|_A = f$.

Definition 2.1 ([4, Definition 3.1]). *A closed subgroup $H \subset G$ is called a large subgroup, if there exists a closed normal subgroup $N \subset G$ such that $N \subset H$ and G/N is a Lie group.*

The large subgroups are characterized in the following result:

Theorem 2.2 ([4, Proposition 3.2]). *Let H be a closed subgroup of a locally compact Hausdorff group G. Then the following conditions are mutually equivalent:*

1. *H is a large subgroup,*
2. *G/H is a metrizable G-ANE(X) for every paracompact proper G-space X.*
3. *G/H is locally contractible.*

Let G be a locally compact group. If Y is a proper G-space, then for every point $y \in Y$ the orbit $G(y)$ is G-homeomorphic to G/G_y (see [Proposition 1.1.5, [11]]). This, in addition with theorem 2.2, yields the following observation:

Observation 2.3. *If Y is a proper G space and there is a point $y \in Y$ such that its isotropy group is a large subgroup, then $G(y)$ is a G-ANE for the class of all paracompact proper G-spaces.*

Definition 2.4 ([4, Definition 3.5]). *A G-space is called a rich G-space if for any point $x \in X$ and any neighborhood $U \subset X$ of x, there exists a point $y \in U$ such that the isotropy group G_y is a large subgroup of G and $G_x \subset G_y$.*

Definition 2.5 ([10]). *Let G be a topologial group, $H \subset G$ a closed subgroup and X a G-space. A subset $S \subset X$ is called an H-slice in X, if:*

1. *S is H-invariant,*

2. *the saturation $G(S)$ is open in X,*

3. *if $g \in G \setminus H$, then $gS \cap S = \emptyset$,*

4. *S is closed in $G(S)$.*

Theorem 2.6 ([4, Definition 3.6])**.** *Let X be a proper G-space and $x \in X$. Then, for any neighborhood U of x in X, there exist a compact large subgroup K of G with $G_x \subset K$, and a K-slice S such that $x \in S \subset U$. Moreover, if X is a rich G-space, then there exists a point $y \in S$ such that $G_y = K$.*

Let X and Y be topological spaces. By a multivalued mapping ϕ from X to Y we understand a map ϕ from X into the non empty sets of Y. By the symbol

$$\phi : X \Rightarrow Y$$

we shall denote that F is a multivalued map from X to Y ([7]).

A multivalued map $\phi : X \Rightarrow Y$ is called lower semicontinuous (l.s.c.) if for each open subset $V \subset Y$, the set

$$\phi^{\Leftarrow}(V) = \{x \in X \mid \phi(x) \cap V \neq \emptyset\}$$

is open in X.

Let X and Y be G-spaces. A multivalued funtion $\phi : X \Rightarrow Y$ will be called equivariant, if

$$\phi(gx) = g\phi(x) = \{gy \mid y \in \phi(x)\},$$

for every $x \in X$ and $g \in G$.

A selection for a multivalued map $\phi : X \Rightarrow Y$ is a continuous mapping $f : X \to Y$ such that $f(x) \in \phi(x)$ for every $x \in X$. If X and Y are G-spaces, a selection $f : X \to Y$ will be an equivariant selection if f is a G-map.

A compatible metric d on a G-space X is called invariant or G-invariant, if $d(gx, gy) = d(x, y)$ for all $g \in G$ and $x, y \in X$.

By a linear G-space we shall mean a real topological vector space on which G acts continuously and linearly, i.e., $g(\lambda x + \mu y) = \lambda(gx) + \mu(gy)$, for every $g \in G$ and for all λ and μ scalars and $x, y \in X$.

We will denote by G-$\mathcal{M}$ the class of all proper G-spaces that admit a G-invariant metric. Let L be a locally convex linear G-space and $Y \subset L$ an invariant convex subset where G acts properly. We will say that (Y, d) belongs to the class G-$\mathcal{L}$ if d is a metric in Y, satisfaying the followings:

1. d is G-invariant,

2. $d(x + z, y + z) = d(x, y)$ for all $x, y \in Y$ and $z \in L$ such that $x + y$ and $x + z$ belong to Y,

3. all open balls determinated by d are convex sets.

If G is compact, it is easy to see that every metrizable convex and invariant subset of any locally convex linear G-space belongs to the class G-$\mathcal{L}$. The same happens for all invariant and convex subsets of any normed linear space where a subgroup of linear isometries acts.

Finally we will denote by G-$\mathcal{P}$ the class of all paracompact proper G-spaces with paracompact orbit space.

By following the proof of [2, Lemma 1] we can infer the next result:

Lemma 2.7. *Let G be a locally compact Hausdorff group and let X be a G space such that $X \in \mathcal{P}$-G. If $\mathcal{U}$ is an open invariant covering of X, then there exists a locally finite open invariant refinement of $\mathcal{U}$.*

In the same way, if we follow the proof of [2, Theorem 1] we can prove the following lemma:

Lemma 2.8. *For any open invariant covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of a proper G-space X such that $X \in G$-$\mathcal{P}$, there exists an invariant partition of unity $\{p_\alpha\}_{\alpha\in\mathcal{A}}$ subordinated to the covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$. That means, $p_\alpha : X \to [0,1]$ is an invariant continuous map, and $p_\alpha^{-1}((0,1]) \subset U_\alpha$, for each $\alpha \in \mathcal{A}$.*

3 A fixed point theorem

Let G be a compact group and let $K \subset L$ a complete convex and invariant subset of a locally convex, metrizable linear G-space, L. By $C(G,K)$ we denote the space of all continuous mappings from G into K, equiped with the compact-open topology. In $C(G,K)$ we can define a continuous action $G \times C(G,K) \to C(G,K)$ as follows:

$$(g,f) \to g * f$$

where $g*f(h) = gf(h)$ for every $h \in G$. For each $f \in C(G,K)$ and $g \in G$ let ${}_gf \in C(G,K)$ be the map definded by the following formula:

$${}_gf(h) = f(gh).$$

Symetrically, we will denote by f_g the continuous map in $C(G,K)$ defined by

$$f_g(h) = f(hg).$$

In [1], the following result is proved which establishes the existence of the vector-valued integral with respect to the Haar measure:

Proposition 3.1 ([1, Lemma 2])**.** *There exists a continuous mapping $\int : C(G,K) \to K$, such that*

(1) $\int {}_gf = \int f = \int f_g$, for all $g \in G$ and $f \in C(G,K)$;

(2) $\int g * f = g \int f$, for all $g \in G$ and $f \in C(G,K)$;

(3) if $f(g) = x_0 \in K$ for every $g \in G$, then $\int f = x_0$.

Corollary 3.2. *Let G be a compact topological group, and let L be a locally convex, metrizable linear G-space. If $K \subset L$ is a G-invariant complete and convex subset, then there exists a point $a \in K$ such that $ga = a$ for all $g \in G$.*

Proof. Pick an arbitrary point $z \in K$ and define $f : G \to K$ as follows:

$$f(g) = gz.$$

Let $\int$ be the map defined in proposition 3.1. We claim that the point $a = \int f \in K$ is the desired point. If g and h are arbitrary elements of G, then we have

$$g * f(h) = gf(h) = ghz = f(gh) = {}_g f(h).$$

So that $g * f = {}_g f$ for each $g \in G$. It follows from proposition 3.1 that

$$ga = g \int f = \int g * f = \int {}_g f = \int f = a$$

for any element $g \in G$. This completes the proof. □

4 Equivariant ε-near selections

Definition 4.1 ([7])**.** *Let (Y, d) be a metric space. Let $F : X \Rightarrow Y$ be a multivalued map and $\varepsilon > 0$. A continuous mapping $f : X \to Y$ is called an ε-near selection if for every $x \in X$,*

$$d(f(x), F(x)) = \inf_{y \in F(x)} d(x, y) < \varepsilon.$$

Definition 4.2. *Let G be a topological group. Let Y be a convex metric subset of a linear space where G acts linearly, and let X be an arbitray G-space. We say that Y has the G-near selection property with respect to X ($Y \in G$-NSP(X)) if every l.s.c. multivalued equivariant map $F : X \Rightarrow Y$ with complete and convex values has, for every $\varepsilon > 0$, an equivariant ε-near selection.*

Theorem 4.3. *Let G be a locally compact Hausdorff group. Let $(Y, d) \in G$-$\mathcal{L}$ and $X \in G$-$\mathcal{P}$. If Y is a rich G-space, then $Y \in G$-NSP(X).*

Before proving theorem 4.3 let us establish the following lemma which is an equivariant version of [7, Lemma 3.2].

Lemma 4.4. *Let G be a locally compact Hausdorff group, $\delta > 0$ and let X and Y be G-spaces. Supose that there exists a compatible metric d on Y such that $(Y, d) \in G$-$\mathcal{M}$. Let $\phi : X \Rightarrow Y$ be a lower semicontinuous multivalued equivariant mapping. In adition, let X_0 be an invariant subset of X for which there exists a continuous equivariant mapping $f : X \to Y$ such that $f|_{X_0}$ is an equivariant δ-near selection for $\phi|_{X_0}$. Then for every $\varepsilon > 0$ there is an invariant neighborhood U_ε of X_0 such that $f|_{U_\varepsilon}$ is an equivariant $\delta + \varepsilon$-near selection.*

Proof. Because d is invariant and ϕ and f are equivariant, it is easy to see that

$$U_\varepsilon = \bigcup_{x \in X_0} f^{-1}(B(f(x_0, \varepsilon/2)) \cap \phi^{\Leftarrow}(B(f(x), \delta + \varepsilon/2))$$

is an invariant neighborhood of X_0. By [7, [Lemma 3.2] the restriction $f|_{U_\varepsilon}$ is a $\delta + \varepsilon$-near selection. □

Proof of Theorem 4.3. Let $\phi : X \Rightarrow Y$ be a l.s.c. multivalued equivariant map with complete and convex values, and let $\varepsilon > 0$. For each $x \in X$, the stabilizer subgroup of x is compact ([11, Proposition 1.1.4]). In addition, because ϕ is equivariant, we have that

$$\phi(x) = \phi(gx) = g\phi(x), \text{ for all } g \in G_x.$$

So, G_x is a compact group acting continuously and linearly in the complete and convex subset, $\phi(x)$. By corollary 3.2, there is a point $a_x \in \phi(x)$ such that $ga_x 0 a_x$ for every $g \in G_x$. Now, the maping $\mu_x : G(x) \to Y$ defined by $\mu_x(ga_x) = ga_x$ is well defined. It is not dificult to see that m_x is an equivariant selection for $\phi|_{G(x)}$.

By theorem 2.6 and since Y is a rich G-space, there exists a point $y_x \in B(a_x, \varepsilon/2) \subset Y$ and there exists a G_{y_x}-slice $S_x \subset B(a_x, \varepsilon/2)$ by y, such that $a_x \in S_x$ and G_{y_x} is a large subgroup containing G_x. Let $r_x : G(S_x) \to G(y_x)$ the equivariant retraction defined by $r_x(gs) = gy_x$ for all $g \in G$ and $s \in S_x$. We can observe that

$$d(r(a_x), a_x) = d(y, a_x) \leq \varepsilon/2.$$

Now we define a new maping $f_x : G(x) \to G(y_x)$ by $f_x(z) = r_x(\mu_x(z))$. Clearly f_x is continuous and equivariant. Therefore,

$$d(f_x(gx), \phi(gx)) \leq d(f_x(gx), ga_x) = d(r_x(\mu_x(gx), ga_x) = d(gy_x, ga_x) = d(y_x, a_x) < \varepsilon/2.$$

Therefore f_x is an equivariant $\varepsilon/2$-near selection for $\phi|_{G(x)}$. By observation 2.3, $G(y_x)$ is a G-ANE for the class of all paracompact proper G-spaces and there exists an invariant neighborhood W_x of $G(x)$ and $F_x : W_x \to Y$ a continuous and equivariant extension of f_x. By lemma 4.4, there exists an invariant neighborhood $U_x \subset W_x$ of $G(x)$ such that $F_x|_{U_x}$ is an equivariant ε-near selection.

Let us do this for every $x \in X$. The family $\{U_x\}_{x \in X}$ is an open invariant covering of X. By lemma 2.7 there exists a locally finite open invariant refinement, $\{O_\alpha\}_{\alpha \in \mathcal{A}}$ of $\{U_x\}_{x \in X}$. For each $\alpha \in \mathcal{A}$, pick a $x(\alpha) \in X$, such that $O_\alpha \subset U_{x(\alpha)}$. Now, for each $\alpha \in \mathcal{A}$, we extend the maping $F_{x(\alpha)}|_{O_\alpha}$ as follows:

$$F_\alpha(z) = \begin{cases} F_{x(\alpha)}(z), & \text{if } z \in O_\alpha, \\ y_0, & \text{if } z \in X \setminus O_\alpha, \end{cases}$$

where y_0 is an arbitrary point in Y. By lemma 2.8, there exists a partition of unity $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ subordinated to $\{O_\alpha\}_{\alpha \in \mathcal{A}}$ such that each $p_\alpha : X \to [0, 1]$ is an invariant map.

The desired ε-near selection $f : X \to Y$ can now be defined by

$$f(x) = \sum_{\alpha \in \mathcal{A}} p_\alpha(x) \tilde{F}_\alpha(x).$$

To see that this works, we observe first that each $x \in X$ has a neighborhood V intersecting only finitely many O_α. In this V, f can be seen as the sum of finitely many continuous maps, and therefore, f is continuous in X. Furthermore, for each $z \in X$, let $Q(z)$ be the subset consiting of all $\alpha \in \mathcal{A}$ such that $z \in O_\alpha$. Since $p_\alpha(z) = 0$ for every $\alpha \notin Q(z)$, then we have

$$f(z) = \sum_{\alpha \in Q(z)} p_\alpha(z) \tilde{F}_\alpha(z).$$

Moreover, if $\alpha \in Q(z)$, then $z \in O_\alpha$, which means that $\tilde{F}_\alpha(z) = F_{x(\alpha)}(z)$. So, we can write $f(z)$ as follows:

$$f(z) = \sum_{\alpha \in Q(z)} p_\alpha(z) F_{x(\alpha)}(z).$$

Since O_α is an invariant subset, we have that $gz \in O_\alpha$ if and only if $z \in O_\alpha$. As a consequence $Q(z) = Q(gz)$, for every $z \in X$ and for all $g \in G$. Now, by using the linearity of the action we observe that

$$\begin{aligned} f(gz) &= \sum_{\alpha \in Q(gz)} p_\alpha(gz) F_{x(\alpha)}(gz) = \sum_{\alpha \in Q(z)} p_\alpha(z) g F_{x_\alpha}(z) \\ &= g \left(\sum_{\alpha \in Q(z)} p_\alpha(z) F_{x(\alpha)}(z) \right) = g f(z). \end{aligned}$$

This proves that f is equivariant. We have still to prove that f is an ε-near selection for $\phi(x)$. To this purpose, we must remember that for every $z \in X$, and for every $\alpha \in Q(z)$, the point $F_{x(\alpha)}(z)$ belongs to the convex set $N_\varepsilon(\phi(z)) = \{y \in Y \mid d(y, \phi(z)) < \varepsilon\}$. So, $f(z)$ is a convex linear combination of finitely many $F_{x(\alpha)}(z)$, all of which lie in the convex set $N_\varepsilon(\phi(z))$, hence $f(z) \in N_\varepsilon(\phi(z))$. This completes the proof of the theorem. □

Theorem 4.5. *Let G be a locally compact Hausdorff group. Let $Y \in G\text{-}\mathcal{L}$ and $X \in G\text{-}\mathcal{P}$. If $Y \in G\text{-ANE}(X)$, then $Y \in G\text{-NSP}(X)$.*

Proof. Copy the prove of theorem 4.3 as far as the construction of the map μ_x. Since Y is a G-ANE(X) we can extend the map μ_x directly to a continuous and equivariant mapping F_x defined on an invariant neighborhood W_x of $G(x)$. Now the proof follows by copying word by word the rest of the proof of theorem 4.3. □

Corollary 4.6. *Let G be a compact group. Let L be a Banach space where G acts continuously and linearly. Thus $L \in G\text{-NSP}(X)$ for every paracompact G-space X.*

Proof. The corollary follows immediately from theorem 4.5 and the following lemma 4.7. □

Lemma 4.7. *Let G be a compact group acting linearly and continuously in a Banach space L. L is a G-ANE(X) for every paracompact G-space X.*

Proof. Let $A \subset X$ be a closed subset of X, and let $f : A \to L$ be a continuous and equivariant map. By [9] L is a ANE(X), meaning that there exists a continuous mapping $F : X \to L$ such that $F|_A = f$. Lets consider now the map $\Phi : X \to C(G, L)$ defined by $\Phi(x)(g) = g^{-1}F(gx)$. The mapping Φ is continuous (see [8, p.95]). Finally we define $\phi(x) = \int \Phi(x)$, where $\int$ is the mapping of proposition 3.1. We claim that ϕ is the desired map. First, ϕ is the composition of two continuous maps, so ϕ is continuous too.

If $a \in A$ then $\Phi(a)(g) = g^{-1}F(ga) = g^{-1}f(ga) = g^{-1}(gf(a)) = f(a)$. That means that $\Phi(a) \in C(G, L)$ is a constant map. By proposition 3.1 we have $\phi(a) = \int \Phi(a) = f(a)$ which proves that $\phi|_A = f$. It remains to prove that ϕ is equivariant. First we observe that

$$\Phi(hx)(g) = g^{-1}F(ghx) = h(gh)^{-1}F(ghx) = h(\Phi(x)(gh)) = (h * \Phi(x))(gh),$$

for every $h, g \in G$ and $x \in X$. Therefore, $\Phi(hx) = (h * \Phi(x))_h$. Finally, by proposition 3.1 we have

$$\phi(hx) = \int \Phi(hx) = \int (h * \Phi(x))_h = \int h * \Phi(x) = h \int \Phi(x) = h\phi(x).$$

This proves that ϕ is equivariant and now the proof is complete. $\square$

Corollary 4.8. *Let G be a compact Lie group. Let L be a locally convex metrizable linear G-space. If $Y \subset L$ is an invariant convex subset, then $Y \in G\text{-NSP}(X)$ for every metrizable G-space X.*

Proof. By [2, Theorem 1] Y is a G-ANE(X). Now the corollary follows directly from theorem 4.5. $\square$

5 Equivariant Selections

Analogously as we have defined the G-near selection property, we can define the *selection property* in the following way:

Definition 5.1. *Let G be a topological group. Let Y be convex metric subset linear space where G acts linearly, and let X be an arbitray G-space. We say that Y has the G selection property respect to X ($Y \in G\text{-SP}(X)$) if every l.s.c. multivalued equivariant map $\phi : X \Rightarrow Y$ with complete and convex values admits an equivariant selection.*

Proposition 5.2. *Let G be a locally compact Hausdorff group. Let $(Y, d) \in G\text{-}\mathcal{L}$. If $Y \in G\text{-NSP}(X)$ for some G-space X, then $Y \in G\text{-SP}(X)$.*

Proof. Let $\phi : X \Rightarrow Y$ be a l.s.c. multivalued equivariant map with complete and convex values. We will construct, by induction, a sequence of continuous and equivariant maps $f_n : X \to Y$ such that, for every $x \in X$,

(a) $d(f_n(x), f_{n+1}(x)) < 2^{-(n-1)}, \quad (n = 1, 2, \dots),$

(b) $d(f_n(x), \phi(x)) < 2^{-n}, \quad (n = 1, 2, \dots).$

Since $Y \in G\text{-NSP}(X)$, there exists an equivariant 1/2-near selection $f_1 : X \to Y$. This map satisfies (b). Suppose that $f_1, \dots, f_n$ have been constructed and satisfy (a) and (b). In order to construct the map f_{n+1}, let us define $\phi_n : X \Rightarrow Y$ as follows:

$$\phi_n(x) = \overline{\phi(x) \cap B(f_n(x), 2^{-n})}.$$

By [12, lemma 1.4.6], ϕ_n is a l.s.c. multivalued map. In addition, for each $x \in X$, $\phi_n(x)$ is a closed subset of the complete set $\phi(x)$, so $\phi_n(x)$ is itself complete. Since the balls defined by the metric d are convex, and since $\phi(x)$ is convex too, we can infer that $\phi(x)$ is a convex subset of Y.

Finally, the invariance of the metric d and the equivariance of the map f_n, tell us that

$$g\phi_n(x) = g(\overline{\phi(x) \cap B(f_n(x), 2^{-n})}) = \overline{g\phi(x) \cap gB(f_n(x), 2^{-n})} = \overline{\phi(gx) \cap B(f_n(gx), 2^{-n})},$$

which means that ϕ_n is equivariant. Now we can aply the fact that $Y \in G$-NSP(X) to find an equivariant $2^{-(n+1)}$-near selection for ϕ_n, let us say $f_{n+1} : X \to Y$. Since $\phi(x) \subset \phi(x)$, we have that

$$d(f_{n+1}(x), \phi(x)) \leq d(f_{n+1}(x), \phi_n(x)) < 2^{-(n+1)}.$$

Then, f_{n+1} satisfies condition (b). In the other hand, $\phi_n(x) \subset \overline{B(f_n(x), 2^{-n})}$. Then

$$d(f_{n+1}(x), f_n(x)) \leq d(f_{n+1}(x), \phi_n(x)) + d(\phi_n(x), f_n(x)) < 2^{-(n+1)} + 2^{-n} < 2^{-n+1},$$

which is (a). This completes the construction by induction.

We claim that $\lim_{n\to\infty} f_n(x)$ exists and belongs to $\phi(x)$, for every $x \in X$. In order to see this, take an arbitrary $x \in X$. By (b), for every $n \in \mathbb{N}$ there exists a point $a_n \in \phi(x)$ such that $d(f_n(x), a_n) < 2^{-n}$. Let us consider the sequence $(a_n)_{n\in\mathbb{N}} \subset \phi(x)$. By (a), we have

$$d(a_n, a_{n+1}) \leq d(a_n, f_n(x)) + d(f_n(x), f_{n+1}(x)) + d(f_{n+1}(x), a_{n+1}(x)) < 2^{-(n-2)}.$$

Therefore $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence contained in the complete subset $\phi(x)$. We conclude that $\lim_{n\to\infty} a_n$ exists and belongs to $\phi(x)$. Since $d(f_n(x), a_n) < 2^{-n}$ for every n, this implies that $\lim_{n\to\infty} f_n(x) = f(x)$ also exists and is equal to $\lim_{n\to\infty} a_n$. This means that $f(x) \in \phi(x)$. By (a), the secuence $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy and thus converges uniformly to f. This implies that f is continuous.

Finally, for every $g \in G$ and $x \in X$, we have

$$f(gx) = \lim_{n\to\infty} f_n(gx) = \lim_{n\to\infty} gf_n(x) = g\left(\lim_{n\to\infty} f_n(x)\right) = gf(x).$$

This proves that f is an equivariant selection for ϕ, meaning that $Y \in G$-SP(x) as we desired.

□

In addition with theorems 4.3 and 4.5 and corollaries 4.6 and 4.8, proposition 5.2 gives us the following and last results:

Corollary 5.3. *Let G be a locally compact Hausdorff group. Let $Y \in G$-$\mathcal{L}$ and $X \in G$-$\mathcal{P}$. If Y is a rich G-space, then $Y \in G$-*SP(X)*.*

Corollary 5.4. *Let G be a locally compact Hausdorff group. Let $Y \in G$-$\mathcal{L}$ and $X \in G$-$\mathcal{P}$. If $Y \in G$-*ANE(X)*, then $Y \in G$-*NSP(X)*.*

Corollary 5.5. *Let G be a compact group. Let L be a linear G-space. If L is a Banach space, then $L \in G$-*SP(X)* for every paracompact G-space X.*

Corollary 5.6. *Let G be a compact Lie group. Let L be a locally convex metrizable linear G-space. If $Y \subset L$ is any invariant convex subset, then $Y \in G$-*SP(X)* for every metrizable G-space X.*

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