Lipschitz subspaces of $C(K)$

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Abstract

Let K be an uncountable metric compact space. It is well known that $C(K)$ is isometrically universal for the separable Banach spaces, but the continuous functions that compose the isometric image of finite dimensional spaces are typically far from being Lipschitz. We prove that the possibility of embedding euclidean spaces $\mathbb{R}^n \hookrightarrow C(K)$ in such a way that the image in $C(K)$ is made of Lipschitz functions is tightly related to the dimension (topological or Hausdorff) of K.

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1. Introduction

Throughout the paper all the Banach spaces considered are real. We shall denote by K a compact Hausdorff space and $C(K)$ will be the Banach space of real continuous functions defined on K endowed with the supremum norm. The real unit interval is denoted by $\mathbb{I}$. We shall consider $\mathbb{I}$ and its finite powers with the euclidean distance. As usual, if X is a Banach space we shall denote by B_X its closed unit ball, and by S_X its unit sphere. For any unexplained concepts or notations about Banach spaces we address the

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reader to [6] or [16].

A classical result of Banach and Mazur [6, Theorem 5.8] says that $C[0,1]$ is isometrically universal for the class of separable Banach spaces. In particular, the euclidean spaces $(\mathbb{R}^n, \|\cdot\|_2)$ can be found isometrically as subsets of functions defined on $[0,1]$. For $n=2$ an isometric embedding $J : \mathbb{R}^2 \to C[0,1]$ can be written explicitly as $J(x_1, x_2)(t) = x_1 \cos(\pi t) + x_2 \sin(\pi t)$, using C^∞ functions. As we see later, an isometric embedding of $\mathbb{R}^3$ cannot be written explicitly using such simple functions. In fact, Peano curves are needed as was first noticed in 1957 by Donoghue [4]. However, $\mathbb{R}^3$ is isometrically embedded into $C([0,1]^2)$ by means of the formula

$$J(x_1, x_2, x_3)(t, s) = x_1 \cos(\pi t)\cos(\pi s) + x_2 \sin(\pi t)\cos(\pi s) + x_3 \sin(\pi s).$$

We will see that the possibility of finding an "easy formula" for an isometric embedding of $\mathbb{R}^n$ into $C(K)$ is related to the *dimension* of K.

If K_1 and K_2 are uncountable metrizable compacta, then $C(K_1)$ and $C(K_2)$ are isomorphic by Milutin's theorem [16, III.D.19]. These Banach spaces cannot be isometric unless K_1 and K_2 are homeomorphic. On the other hand, $C(K_1)$ and $C(K_2)$ are universal spaces for the class of separable spaces in the isometric category. In particular $C(K_1)$ contains an isometric copy of $C(K_2)$ and vice versa. In particular, that means that it is not possible to distinguish between K_1 and K_2 by isometric embeddings of test spaces.

Our idea is to relate properties of a compact K to the existence of isometric embeddings $J : X \to C(K)$ of finite dimensional linear spaces X such that the set $J(X)$ is composed of "nice" functions. Here nice will mean Lipschitz at least, and the requirement of finite dimension is necessary. Indeed, it is easy to see that if the isometric embedding $J(X)$ is composed of Lipschitz functions, then X must be of finite dimension (Proposition 2.1). The next result shows the relation between K and the existence of nice embeddings of $\mathbb{R}^n$.

Theorem 1.1. *Let (K,d) be an uncountable metric compact space and $n \in \mathbb{N}$. The following are equivalent:*

(i) There is an onto Lipschitz mapping $\phi : K \to \mathbb{I}^n$.

(ii) $C(K)$ *contains an isometric copy of any* $(n+1)$*-dimensional Banach space made of Lipschitz functions.*

(iii) $C(K)$ *contains an isometric copy of the euclidean space* $(\mathbb{R}^{n+1}, \|\cdot\|_2)$ *made of Lipschitz functions.*

Moreover, if K *has a compatible structure of Lipschitz manifold, then statements (i), (ii) and (iii) are also equivalent to*

(iv) $dim(K) \geq n$.

We follow [11] for the definition of Lipschitz manifold (with boundary). Compatible structure of Lipschitz manifold guarantees that the charts are bi-Lipschitz between the global metric d and the local metric. Sullivan [14] proved that n-dimensional topological manifolds have a Lipschitz structure for $n \neq 4$. In any case, we need to fix a metric on K since topologically equivalent metrics on K are in general not Lipschitz equivalent. If K is not a Lipschitz manifold, we may still obtain information about K from the previous result using other notions of dimension. If we consider the Hausdorff dimension on K, then statement (i) clearly implies that the Hausdorff dimension of K is greater or equal than n (see [7, Corollary 2.4]). On the other hand, a recent result of Keleti, Máthé and Zindulka [9] says that if the Hausdorff dimension of K is strictly greater than n, then statement (i) holds. Unfortunately, the existence of a Lipschitz mapping onto a cube does not characterize the Hausdorff dimension as showed by the example constructed by Vituškin, Ivanov and Melnikov [15]. If K is ultrametric, then statement (i) implies that the Hausdorff dimension is at least n by another result of [9].

In the case of smooth manifolds, the regularity of the functions composing the isometric copy of the euclidean space is as good as possible.

Theorem 1.2. *Let* K *be a compact* C^r*-manifold for* $r = 1, \dots, \infty$ *of dimension* $n-1$ *for* $n \geq 2$. *Then*

(a) $C(K)$ *contains an isometric copy of* $(\mathbb{R}^n, \|\cdot\|_2)$ *made of* C^r*-smooth functions;*

(b) $C(K)$ *contains no isometric copy of* $(\mathbb{R}^{n+1}, \|\cdot\|_2)$ *made of* C^1*-smooth functions or Lipschitz functions.*

We have chosen the euclidean space as test space because of its easiness, but any finite dimensional space with strictly convex dual will work as a test space. On the other hand, polyhedral spaces can always be isometrically embedded using nice functions. Recall that a finite dimensional Banach space is *polyhedral* if its unit ball is a convex polytope.

Theorem 1.3. *If K is an infinite metric compact space, then $C(K)$ contains isometric copies made of Lipschitz functions of any finite dimensional polyhedral space.*

The proofs of these results depend on some easy facts about Lipschitz maps, Lipschitz manifolds and isometric embeddings into $C(K)$ spaces that we will develop in the next section. We finish the paper with some remarks about extending the results for Hölder maps and the typical n-dimensional subspaces of $C(K)$.

2. Auxiliary results

We denote by $L(K,d)$ the Lipschitz functions (with respect to d) of $C(K)$. The Lipschitz constant for $f \in L(K,d)$ is the number

$$L(f) = \sup\left\{ \frac{|f(t_1) - f(t_2)|}{d(t_1,t_2)} : t_1, t_2 \in K, t_1 \neq t_2 \right\}.$$

Proposition 2.1. *Let $X \subset C(K)$ be a non trivial linear subspace. Then either*

(a) $X \cap L(K,d)$ is of first category in X;

(b) or $X \subset L(K,d)$, X is finite dimensional and there exists $\lambda > 0$ such that $L(f) \leq \lambda \|f\|$ for every $f \in X$.

Proof. Observe that $X \cap L(K,d) = \bigcup_{n=1}^{\infty} \{f \in X : L(f) \leq n\}$ is a decomposition into countably many closed balanced convex sets. If $X \cap L(K,d)$ is not of first category in X, then there is $\delta > 0$ such that $\delta B_X \subset \{f \in X : L(f) \leq n\}$ for some $n \in \mathbb{N}$. By homogeneity, we have $L(f) \leq \lambda \|f\|$ with $\lambda = \delta^{-1} n$ for every $f \in X$. In particular $X \subset L(K,d)$. Note that B_X is a complete, bounded and equicontinuous set of functions, and hence it is compact by Ascoli's theorem [10]. Therefore X must be of finite dimension. □

Recall that given a linear operator $T : X \to Y$ between Banach spaces, the *adjoint operator* $T : Y^* \to X^*$ is the linear map defined by the rule

$$T^*(y^*)(x) = y^*(T(x)).$$

If T is bounded then T^* is also bounded and $\|T\| = \|T^*\|$.

On the other hand, observe that K is naturally embedded in $C(K)^*$ if we define $k : C(K) \to \mathbb{R}$ as $k(f) := f(k)$ for every $k \in K$. In this case, we always have that $K \subset B_{C(K)^*}$.

Proposition 2.2. *Let $J : X \to C(K)$ be an isomorphic embedding. Then $J(X) \subset L(K, d)$ if and only if $J^*|_K$ is Lipschitz from d to the norm of X^*, where J^* denotes the adjoint map from $C(K)^*$ into X^*.*

Proof. If J^* is Lipschitz, then any function $J(x)$ is Lipschitz as well, since $J(x)(t) = J^*(t)(x)$. Reciprocally, assume that $J(X) \subset L(K, d)$. By Proposition 2.1 there is $\lambda > 0$ such that $L(f) \leq \lambda$ for every $f \in J(X)$. Now, if $x \in B_X$ and $t_1, t_2 \in K$ then

$$|J^*(t_1)(x) - J^*(t_2)(x)| = |J(x)(t_1) - J(x)(t_2)| \leq \lambda\, d(t_1, t_2).$$

Taking supremum on $x \in B_X$ we get $\|J^*(t_1) - J^*(t_2)\| \leq \lambda\, d(t_1, t_2)$. $\square$

Remark 2.3. *A function $f : K \to \mathbb{R}$ is said to be α-Hölder for $\alpha \in (0, 1]$ if there is a constant $\lambda > 0$ such that $|f(x) - f(y)| \leq \lambda\, d(x, y)^\alpha$ for any $x, y \in K$. It is easy to see that the Lemma 2.1 and Proposition 2.2 can be generalized to the setting of α-Hölder functions.*

The set of extreme points of a convex set C is denoted by $\mathrm{Ext}(C)$.

Proposition 2.4. *Let X be a Banach space and let $J : X \to C(K)$ be a linear operator with $\|J\| \leq 1$. Then J is an isometric embedding if and only if*

$$\mathrm{Ext}(B_{X^*}) \subset J^*(K) \cup (-J^*(K)).$$

Proof. Note that, in general, $J : X \to Y$ is an isometric embedding if and only if $J^*(B_{Y^*}) = B_{X^*}$ (the less easy part relies on the Hahn–Banach theorem). Hence we have just to check that the statement above is equivalent to $B_{X^*} = J^*(B_{C(K)^*})$. Clearly, $\|J^*\| = \|J\| \leq 1$ implies that $J^*(B_{C(K)^*}) \subset B_{X^*}$. On the other hand, if $\mathrm{Ext}(B_{X^*}) \subset J^*(B_{C(K)^*})$, then $B_{X^*} \subset J^*(B_{C(K)^*})$ by

the Krein–Milman Theorem [6, Theorem 3.65].
For the converse implication, observe that

$$J^*(B_{C(K)^*}) = J^*(\overline{\text{conv}}^{w^*}(K \cup (-K))) = \overline{\text{conv}}^{w^*}(J^*(K \cup (-K)))$$

by the weak* continuity of J^*. Therefore, if $B_{X^*} = J^*(B_{C(K)^*})$, and then $\text{Ext}(B_{X^*}) \subset J^*(K \cup (-K))$ by Milman's Theorem [6, Theorem 3.66]. $\square$

Corollary 2.5. *Let X be a Banach space. There exists an isometric embedding of X into $C(K)$ if and only if there exists a continuous mapping $\Psi : K \to B_{X^*}$ such that*

$$\text{Ext}(B_{X^*}) \subset \Psi(K) \cup (-\Psi(K)).$$

Proof. If such an isometric embedding $J : X \to C(K)$ exists, then $\Psi = J^*|_K$. For the other implication, define $J(x)(t) = \Psi(t)(x)$. Evidently, J is a linear operator with $\|J\| \leq 1$ that satisfies $J^*|_K = \Psi$. So, by Proposition 2.4, it is an isometric embedding. $\square$

The first part of the following result is due to Donoghue [4] who used it for the construction of Peano-type filling curves.

Corollary 2.6. *Let X be a Banach space such that X^* is strictly convex and let $J : X \to C(K)$ be an isometric embedding. Then*

$$S_{X^*} \subset J^*(K) \cup (-J^*(K)).$$

Moreover, there exists $t_1, t_2 \in K$ such that $J(x)(t_1) = -J(x)(t_2)$ for every $x \in X$.

Recall that a Banach space X is *strictly convex* if given $x, y \in S_X$, with $x \neq y$ then $\|\frac{x+y}{2}\| < 1$.

Proof. In this case $\text{Ext}(B_{X^*}) = S_{X^*}$. Therefore $S_{X^*} \subset J^*(K) \cup (-J^*(K))$. But the connection of S_{X^*} implies that $J^*(K) \cap (-J^*(K)) \neq \emptyset$. Take $x^* \in J^*(K) \cap (-J^*(K))$ and $t_1, t_2 \in K$ such that $J^*(t_1) = x^*$ and $J^*(t_2) = -x^*$. Thus, for every $x \in X$ we have that $J(x)(t_1) = J^*(t_1)(x) = -J^*(t_2)(x) = -J(x)(t_2)$, as desired. $\square$

Before proving the main theorem, let us establish the following easy lemma

Lemma 2.7. *Let K be a k-dimensional Lipschitz manifold. Then $k \geq n$ if and only if there exists a Lipschitz onto map $f : K \to \mathbb{I}^n$.*

Proof. Since K is a k-dimensional Lipschitz manifold, we can find a finite cover $\{O_1, O_2, \dots, O_m\}$ of K each member of which is bi-Lipschitz homeomorphic to $\mathbb{I}^k$. In particular, there exists a Lipschitz onto map $\phi : O_1 \to \mathbb{I}^k$. Since $\mathbb{I}^k$ is an absolute Lipschitz retract, by [2, Proposition 1.2] we can find a Lipschitz (onto) extension $F : K \to \mathbb{I}^k$ of ϕ. Now, if $k \geq n$, the projection in the first n-coordinates $p : \mathbb{I}^k \to \mathbb{I}^n$ is a Lipschitz map with Lipschitz constant 1. Therefore, the composition $p \circ F : K \to \mathbb{I}^n$ is a Lipschitz map, which proves the first implication.

Now assume that $f : K \to \mathbb{I}^n$ is a Lipschitz onto map. Since $\{f(O_i)\}_{i=1}^m$ is a cover of $\mathbb{I}^n$ by compact sets, there must exist $i \in \{1, \dots, m\}$ such that $f(O_i)$ has non-empty interior in $\mathbb{I}^n$. Thus, we can find a closed cube $B \subset f(O_i)$ Lipschitz homeomorphic to $\mathbb{I}^n$. Therefore B is an absolute Lipschitz retract, and then we can find a Lipschitz retraction $r : f(O_i) \to B$. Finally observe that the composition $r \circ f|_{O_i} : O_i \to B$ is a Lipschitz map and this can only be possible if $k \leq n$. $\square$

3. Proofs of the main results and final remarks

Proof of Theorem 1.1. (i)$\Rightarrow$(ii) If X is $(n+1)$-dimensional, the dual sphere S_{X^*} is a Lipschitz manifold of dimension n. There exists a Lipschitz mapping $\psi : \mathbb{I}^n \to S_{X^*}$ such that $S_{X^*} \subset \psi(\mathbb{I}^n) \cup (-\psi(\mathbb{I}^n))$. For $x \in X$, consider the function $J(x) \in C(K)$ given by $J(x)(t) = \psi(\phi(t))(x)$. Clearly, $J^*(K) = \psi(\mathbb{I}^n)$, and so J is an isometric embedding by Corollary 2.5.

(ii)$\Rightarrow$(iii) is trivial. (iii)$\Rightarrow$(i) Assume that $X = (\mathbb{R}^{n+1}, \|\cdot\|)$ embeds into $C(K)$ with an isometric embedding J. If $J(X) \subset L(K, d)$, then J^* is Lipschitz by Proposition 2.1. Now, by Corollary 2.6 we have

$$S_{X^*} \subset J^*(K) \cup (-J^*(K)).$$

That implies the existence of a point $x^* \in S_{X^*}$ and $\delta > 0$ such that $S_{X^*} \cap B[x^*, \delta] \subset J^*(K)$. We may suppose that δ is small enough to guarantee that $S_{X^*} \cap B[x^*, \delta]$ is Lipschitz homeomorphic to $\mathbb{I}^n$. Therefore $S_{X^*} \cap B[x^*, \delta]$ is an absolute Lipschitz retract. If ψ is a Lipschitz retraction of B_{X^*} onto $S_{X^*} \cap B[x^*, \delta]$, then $\psi \circ J^*$ is a Lipschitz mapping from K onto $S_{X^*} \cap B[x^*, \delta]$ which is Lipschitz homeomorphic to $\mathbb{I}^n$. This proves the desired implication.

Finally if K is a Lipschitz manifold, implication $(i) \Leftrightarrow (iv)$ follows immediately from Lemma 2.7. $\square$

Proof of Theorem 1.3. If X is polyhedral and finite dimensional, its dual X^* is also polyhedral and so $\mathrm{Ext}(B_{X^*}) = \{x_1^*, \dots, x_N^*\}$ is a finite set. Take different points $\{t_n\}_{n=1}^N \subset K$ and disjointly supported Lipschitz functions $\psi_n : K \to [0,1]$ such that $\psi_n(t_m) = 0$ if $n \neq m$ and $\psi_n(t_n) = 1$. The mapping $\Psi : K \to B_{X^*}$ given by $\Psi(t) = \sum_{n=1}^N \psi_n(t) x_n^*$ is well defined and Lipschitz. The linear operator $J : X \to C(K)$ defined by $J(x)(t) = \Psi(t)(x)$ satisfies that $\|J\| \leq 1$ and $J^*(t) = \Psi(t)$ for every $t \in K$, and thus $\mathrm{Ext}(B_{X^*}) \subset J^*(K)$. By Corollary 2.5, J is an isometric embedding. $\square$

Proof of Theorem 1.2. Denote $\mathbb{R}^n$ with the euclidean norm by X. If K is a $(n-1)$-dimensional C^r-manifold, then there is $H \subset K$ compact which is C^r-homeomorphic to $\mathbb{I}^{n-1}$ and such that there exists a C^r-smooth retraction $\psi : K \to H$. Find a C^∞-smooth mapping $\phi : H \to S_{X^*}$ such that $S_{X^*} \subset \phi(H) \cup (-\phi(H))$. Define $J : X \to C(K)$ by $J(x)(t) = \phi(\psi(t))(x)$. Clearly $J(x)$ is a C^r-smooth function and J is an isometric embedding by Corollary 2.5, and completes the proof of (a).
If $C(K)$ contains an isometric copy of $(\mathbb{R}^{n+1}, \|\cdot\|)$ made of Lipschitz functions (in particular, if they are C^1-smooth), then there is a Lipschitz mapping of K onto $\mathbb{I}^n$, and so $\dim(K) \geq n$. $\square$

We will finish with some remarks:

(1) There exist Peano's filling curves in the Hölder class, see [13, Theorem 3.1] for instance, where it is shown that there is an $1/2$-Hölder surjection from $\mathbb{I}$ onto $\mathbb{I}^2$. Note that such a map with small modifications and the help of Corollary 2.5 provides an embedding of $(\mathbb{R}^3, \|\cdot\|_2)$ into $C[0,1]$ made of $1/2$-Hölder functions. Indeed, if $\phi : \mathbb{I} \to \mathbb{I}^2$ is onto and $1/2$-Hölder, write $\phi = (\tau, \sigma)$ and note that $J : \mathbb{R}^3 \hookrightarrow C[0,1]$ defined by

$$J(x_1, x_2, x_3) = x_1 \cos(\pi\tau)\cos(\pi\sigma) + x_2 \sin(\pi\tau)\cos(\pi\sigma) + x_3 \sin(\pi\sigma)$$

is an isometric embedding made of $1/2$-Hölder functions.

(2) Hausdorff dimension has a good behavior under Hölder maps, [7, Proposition 2.3]. Therefore it would be possible to obtain information about the Hausdorff dimension of a metric compact K from the dimension of its Hölder euclidean subspaces, as in Theorem 1.1. This observation combined with Remark 2.3 and the ideas from the previous remark can be used to generalize Theorem 1.1 in Hölder case.

(3) In the remaining remarks we will use the techniques about isometric embeddings into $C(K)$ spaces to understand how a "typical" n-dimensional subspace of $C(K)$ looks like. Let us start by recalling that the strictly convex norms are generic in the following sense: the set of strictly convex norms on a separable Banach space is a dense $\mathcal{G}_\delta$-set in the metric space of equivalent norms endowed with the Banach-Mazur distance. In particular, the "generic norm" on $\mathbb{R}^n$ is strictly convex and smooth. Baire category theorem allows us to blend generic properties of norms, see the Asplund averaging technique [3, p. 52].

(4) However, a "typical" n-dimensional subspace of $C[0,1]$ is far from being smooth. A subspace of $C(K)$ of dimension equal or less than n is determined by n "random" functions $\{f_1, \dots, f_n\} \subset C(K)$. Putting $F = (f_1, \dots, f_n)$ this is an element $F \in C(K, \mathbb{R}^n)$, and name $X_F = \operatorname{span}\{f_1, \dots, f_n\}$. Let J_F be the mapping from $\mathbb{R}^n$ into $C(K)$ given by

$$J_F(x_1, \dots, x_n) = x_1 f_1 + \cdots + x_n f_n$$

and endow $\mathbb{R}^n$ with the seminorm $p_F(x_1, \dots, x_n) = \|x_1 f_1 + \cdots + x_n f_n\|_\infty$. If X_F has dimension n, then J_F is the isometric embedding of $(\mathbb{R}^n, p_F)$ into $C(K)$. Clearly, we have $J_F^*|_K = F$. Suppose that X_F has strictly convex dual, then the radial boundary of $F(K) \cup (-F(K))$ should be a $(n-1)$-dimensional sphere, by Corollary 2.6. This seems to be "highly unlikely". Indeed, in [1] the authors proved that from a generic point of view the Hausdorff dimension of $F(K)$ for $F \in C(K, \mathbb{R}^n)$ is the minimum of n and the topological dimension of K. In particular, if $K = [0,1]$, the set $F(K)$ has generic Hausdorff dimension 1. That implies for $n > 2$ that X_F^* is generically far from being strictly convex.

(5) Finally, we may compare two n-dimensional subspaces of $C(K)$ by measuring the Hausdorff distance $d_{\mathcal{H}}$ between their unit balls (or spheres). Recall that this way of measuring the distance between subspaces of a Banach space was introduced by Kadets in [8]. Following the notation above, the next two observations show that the relation between F and X_F is continuous back and forth (we omit the elementary proofs)

(a) Given $F \in C(K, \mathbb{R}^n)$ such that X_F is n-dimensional and $\varepsilon > 0$, there is $\delta > 0$ such that for any $G \in C(K, \mathbb{R}^n)$ with $\|F - G\|_\infty < \delta$, then $d_{\mathcal{H}}(X_F, X_G) < \varepsilon$.

(b) Given $F \in C(K, \mathbb{R}^n)$ and $\varepsilon > 0$, there is $\delta > 0$ such that if $X \subset C(K)$ satisfies that $d_{\mathcal{H}}(X_F, X) < \delta$, we can then find $G \in C(K, \mathbb{R}^n)$ with $X = X_G$ and $\|F - G\|_\infty < \varepsilon$.

Now, consider the space of n-dimensional subspaces of $C[0,1]$, equipped with the topology induced by $d_{\mathcal{H}}$ (in the sense of Kadets). In this case, polyhedral subspaces are dense, however smooth subspaces are not dense for $n \geq 2$. Indeed, given $\varepsilon > 0$ and X_F a n-dimensional subspace of $C[0,1]$, by observation (a) we can find $G \in C(\mathbb{I}, \mathbb{R}^n)$ close enough to F in order that $d_{\mathcal{H}}(X_F, X_G) < \varepsilon$ and such that $\overline{conv}(G(\mathbb{I}))$ has finitely many extreme points. This implies that X_G^* is polyhedral and thus X_G is polyhedral too. To prove the other statement consider a subspace X_F such that $F(\mathbb{I})$ is far from its convex hull (for example if $F(\mathbb{I})$ is a star). Fix $\varepsilon > 0$ such that $(F(\mathbb{I}) \cup (-F(\mathbb{I}))) + \varepsilon B_{\mathbb{R}^n}$ is not convex. Any subspace $X \subset C(\mathbb{I})$ close enough to X_F is of the form $X = X_G$ with $\|G - F\|_\infty < \varepsilon$ by observation (b). With such a choice, $G(\mathbb{I}) \cup (-G(\mathbb{I})) \subset (F(\mathbb{I}) \cup (-F(\mathbb{I}))) + \varepsilon B_{\mathbb{R}^n}$ is far from containing the boundary of a large convex body, and so $B_{X^*} = \overline{conv}(G(\mathbb{I}) \cup (-G(\mathbb{I})))$ cannot be strictly convex. Therefore X is not smooth by Šmulyan's duality [3, Proposition 1.6] or [6, Corollary 7.23].

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