Spacelike hypersurfaces with a canonical principal direction

Dedicated to Franki Dillen in memoriam

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Abstract

We give a characterization of spacelike hypersurfaces with a canonical principal direction induced by a timelike closed and conformal vector field on a Lorentzian manifold. As an application we prove that a maximal spacelike surface in a Lorentzian 3-manifold making a constant hyperbolic angle with a parallel vector field should be totally geodesic and flat.

1 Introduction

Given a vector field X in a Lorentzian manifold $\bar{M}$, a spacelike hypersurface M is said to have a canonical principal direction relative to a timelike vector field Z if the projection of Z onto the tangent space of the hypersurface gives a principal direction. This concept was introduced by Dillen in [1]. Riemannian examples of these hypersurfaces include classic objects as the Archimedes spiral, cones and cylinders. Of course, for such a hypersurface to have nice properties, it is natural to impose some condition on the vector field Z which defines it; parallel, Killing or conformal vector fields are some options to consider. In this paper we impose the condition on Z for it to be closed conformal; see equation (1) below. As it will turn out, this setting allows us to give several characterizations of the canonical principal direction spacelike hypersurfaces. To do this we introduce the hyperbolic angle funtion θ between Z and a unit timelike vector ξ normal to M.

This paper is organized as follows. After some preliminaries and calculations which may be interesting for further study, we characterize the spacelike hypersurfaces with a canonical principal direction in our main result, Theorem 2.8. To give the statement, let T be the unit vector field defined as the projection of Z onto the tangent space of M.

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The following statements are equivalent:

1. *M has a canonical principal direction relative to Z, i.e., T is a principal direction.*
2. *The angle θ between Z and ξ is constant along the directions tangent to M and orthogonal to T.*

In addition, if we consider an open subset of $\bar{M}$ isometric to a warped product $-I \times_{\varrho} N$ where M is the graph of a function F, the above conditions are equivalent to the following:

3. *The integral curves of T are geodesics in M.*
4. *The norm of the gradient of the height function $h(t, x) = t$ is constant along the level curves of h.*
5. *The norm of the gradient of F is constant along the level curves of F.*

In the second part of this paper we consider the special case when Z is parallel and the hyperbolic angle function θ is constant. Moreover, in our final result, Corollary 2.11, we characterize the maximal surfaces of this kind, as follows:

Let M be a spacelike surface in a three dimensional Lorentzian manifold $\bar{M}^3$. Let us assume that M has constant hyperbolic angle function θ relative to a timelike parallel vector field Z on $\bar{M}$. If M is maximal then it is totally geodesic and if the constant angle is nonzero then it is also flat.

More results in the case of Lorentzian ambients can be found in [4] and [2].

2 Preliminaries

In order to do some calculations, we will need the concept of hyperbolic angle between timelike vectors, which we include here for completeness.

Proposition 2.1. (See [6] page 144) *Let u and v be timelike vectors in a Lorentzian vector space V. We say that u, v are in the same timecone if $\langle u, v\rangle < 0$. Let us denote the norm of a vector by $|u| := |\langle u, u\rangle|^{1/2}$. Then*

1. *$|\langle u, v\rangle| \geq |u||v|$, with equality if and only if u and v are collinear.*
2. *If u, v are in the same timecone of V, there is a unique number $\theta \geq 0$, called the hyperbolic angle between u and v, such that $\langle u, v\rangle = -|u||v| \cosh\theta$.*

Here we will work in the Lorentzian setting. The vector fields which we will distinguish are given in the following definition.

Definition 2.2. Let $\bar{M}$ be a Lorentzian manifold. A vector field $Z \in \mathfrak{X}(\bar{M})$ is *closed conformal* if and only if

$$\bar{\nabla}_Y Z = \varphi Y \tag{1}$$

for every $Y \in \mathfrak{X}(\bar{M})$, where φ is a differentiable function defined on $\bar{M}$.

Closed conformal vector fields have been studied extensively; see [5], in particular, where S. Montiel proved the following interesting facts which we use freely in this paper:

Theorem 2.3. *Let $\bar{M}^{n+1}$ be a Lorentzian manifold endowed with a timelike closed conformal vector field Z satisfying (1). Then,*

- *Z defines a n-dimensional distribution $Z^{\perp}$ by taking at each point the orthogonal complement of Z. This distribution is integrable and each leaf of the corresponding foliation is totally umbilical in $\bar{M}$.*

- *The functions $|Z|$ and φ are constant along each leaf of the aforementioned foliation.*

- *Fix a connected component N of a leaf of the foliation and let ψ_t be the local flow of Z, defined in an open interval $I \subset \mathbb{R}$. Then the expression*

$$\varrho(t) = |Z_{\psi_t(p)}|, \quad p \in N,$$

does not depend on the particular value chosen for p and $\bar{M}$ is locally isometric to the Lorentzian warped product $-I \times_{\varrho} N$. From this form we may recover the closed conformal vector field Z as

$$Z = |Z|\,\partial_t = \varrho\,\partial_t,$$

where ∂_t is the lift to $\bar{M}$ of the canonical vector field tangent to I.

Definition 2.4. Let $\bar{M}$ be a Lorentzian manifold and assume that M is a spacelike hypersurface of $\bar{M}$, so that there exists a unit timelike vector field $\xi : M \to TM^{\perp}$. Let Z be a closed conformal vector field on $\bar{M}$ such that $Z_{|M}$ is timelike and lies in the timecone of ξ at each point of M, that is, $\langle Z(p), \xi(p)\rangle < 0$ at each $p \in M$. Let $\theta(p)$ measure the hyperbolic angle between $Z(p)$ and $\xi(p)$. Proposition 2.1 implies that for every $p \in M$,

$$\cosh\theta(p) = -\langle Z(p)/|Z(p)|, \xi(p)\rangle.$$

Now we will use θ to give a decomposition of Z: We define $T = Z^T/|Z^T|$, where Z^T denotes the tangent part of Z in TM. Similarly, $Z^{\perp}$ denotes the component of Z in $TM^{\perp}$ and then along M we have the decomposition $Z = Z^T + Z^{\perp}$.
Since M has codimension one and ξ is timelike, we have the formula $Z^{\perp}(p) = -\langle Z(p), \xi(p)\rangle\xi(p)$. Therefore, $Z(p) = Z^T(p) - \langle Z(p), \xi(p)\rangle\xi(p) = Z^T(p) + |Z(p)|\cosh\theta(p)\xi(p)$. From now, we will omit p in our notation. In particular, we deduce that

$$\langle Z^T, Z^T\rangle = \langle Z, Z\rangle + |Z|^2\cosh^2\theta = -\langle Z, Z\rangle\sinh^2\theta.$$

Here, we used that $|Z|^2 = |\langle Z, Z\rangle| = -\langle Z, Z\rangle$ because $Z_{|M}$ is timelike. Then $|Z^T| = |Z|\sinh\theta$. Finally, we obtain that

$$Z^T = \langle Z, Z^T/|Z^T|\rangle Z^T/|Z^T| = \langle Z^T, Z^T/|Z^T|\rangle T = |Z^T|T.$$

This proves that

$$Z/|Z| = \sinh\theta\, T + \cosh\theta\,\xi. \tag{2}$$

Remark 2.5. Let us observe that $\sinh\theta = 0$ if and only if $\theta = 0$ and in this case $\cosh\theta = 1$. So, if $\sinh\theta = 0$ then $Z/|Z| = \xi$, i.e. Z and ξ are collinear. The converse of this implication follows from Proposition 2.1.

Proposition 2.6. *Let M be a spacelike hypersurface in $\bar{M}$ and let Z be a closed conformal vector field Z on $\bar{M}$ such that $Z_{|M}$ is timelike. Then for each vector field Y tangent to M we have*

$$\varphi Y/|Z| + \varphi\sinh^2\theta\frac{\langle Y, T\rangle}{|Z|}T = (Y\cdot\theta)\cosh\theta T + \sinh\theta\nabla_Y T - \cosh\theta A^{\xi}(Y). \tag{3}$$

and

$$\alpha(Y, T) + (Y\cdot\theta)\xi = \varphi\cosh\theta\frac{\langle Y, T\rangle}{|Z|}\xi. \tag{4}$$

Proof. Since Z is closed conformal, $Y \cdot \langle Z, Z\rangle = 2\varphi\langle Y, Z\rangle$. Now,

$$2|Z|(Y \cdot |Z|) = Y \cdot |Z|^2 = -Y \cdot \langle Z, Z\rangle = -2\varphi\langle Y, Z\rangle.$$

Then, $Y \cdot |Z| = -\varphi\langle Y, Z/|Z|\rangle$. Finally, $Y \cdot |Z|^{-1} = -(Y \cdot |Z|)/|Z|^2 = \varphi\langle Y, Z\rangle/|Z|^3$. Applying equation (2), $Y \cdot |Z|^{-1} = \varphi \sinh\theta\langle Y, T\rangle/|Z|^2$.
We are ready for the main calculations. Let us take the derivative on each side of equation (2), obtaining

$$\bar{\nabla}_Y(Z/|Z|) = \varphi Y/|Z| + \varphi \sinh\theta\frac{\langle Y, T\rangle}{|Z|}(Z/|Z|)$$

and

$$\bar{\nabla}_Y(\sinh\theta T + \cosh\theta\xi) = (Y \cdot \theta)\cosh\theta T + \sinh\theta\,(\nabla_Y T + \alpha(Y, T)) + (Y \cdot \theta)\sinh\theta\xi - \cosh\theta A^\xi(Y).$$

By equating the tangent and normal parts of the above, we have

$$\begin{aligned}
\varphi Y/|Z| + \varphi \sinh^2\theta\frac{\langle Y, T\rangle}{|Z|}T &= (Y \cdot \theta)\cosh\theta T + \sinh\theta\nabla_Y T - \cosh\theta A^\xi(Y).\\
\varphi \sinh\theta\cosh\theta\frac{\langle Y, T\rangle}{|Z|}\xi &= \sinh\theta\alpha(Y, T) + (Y \cdot \theta)\sinh\theta\xi. \quad \square
\end{aligned}$$

Corollary 2.7. *Let W be a section of TM so that T, W are orthogonal. Then*

$$A^\xi(T) = (T \cdot \theta)T + \tanh\theta\nabla_T T - \frac{\varphi}{|Z|}\cosh\theta T, \tag{5}$$

$$A^\xi(W) = (W \cdot \theta)T + \tanh\theta\nabla_W T - \frac{\varphi}{|Z|}\ \mathit{sech}\ \theta W, \tag{6}$$

$$\alpha(T, W) = -(W \cdot \theta)\xi, \quad \alpha(T, T) = -(T \cdot \theta)\xi + \frac{\varphi}{|Z|}\cosh\theta\xi, \tag{7}$$

$$W \cdot \theta = \tanh\theta\langle\nabla_T T, W\rangle. \tag{8}$$

Proof. If we take $Y = W$ in equation (3), $\varphi W/|Z| = (W \cdot \theta)\cosh\theta T + \sinh\theta\nabla_W T - \cosh\theta A^\xi(W)$. If we take $Y = W$ in equation (4), $\alpha(W, T) = -(W \cdot \theta)\xi$. Let us observe that the first equation implies the second one.
This in turn implies that

$$A^\xi(W) = \frac{1}{\cosh\theta}(\sinh\theta\nabla_W T - \frac{\varphi}{|Z|}W) + (W \cdot \theta)T.$$

If we take $Y = T$ in equation (4),

$$\alpha(T, T) = -(T \cdot \theta)\xi + \varphi\cosh\theta\frac{1}{|Z|}\xi.$$

If we take $Y = T$ in equation (3),

$$\varphi T/|Z| + \varphi \sinh^2\theta\frac{1}{|Z|}T = (T \cdot \theta)\cosh\theta T + \sinh\theta\nabla_T T - \cosh\theta A^\xi(T).$$

So that

$$A^\xi(T) = (T \cdot \theta)T + \frac{1}{\cosh\theta}(\sinh\theta\nabla_T T - \varphi T/|Z| - \varphi\sinh^2\theta\frac{1}{|Z|}T)$$

and from this we have that

$$\langle\alpha(T, T), \xi\rangle = (T \cdot \theta) + \frac{1}{\cosh\theta}(-\varphi/|Z| - \varphi\sinh^2\theta\frac{1}{|Z|}) = (T \cdot \theta) - \frac{1}{\cosh\theta}\varphi\cosh^2\theta\frac{1}{|Z|}$$

which is the same information we obtained with equation (4) above.
Another consequence of equation (3) is that $\langle \alpha(T, W), \xi \rangle = \tanh\theta \langle \nabla_T T, W \rangle$. Therefore,

$$\alpha(T, W) = -\tanh\theta \langle \nabla_T T, W \rangle \xi,$$

because ξ is timelike. Using the above information, we conclude

$$W \cdot \theta = \tanh\theta \langle \nabla_T T, W \rangle. \quad \square$$

We fix some notation before proving our main result, giving different characterizations of the hypersurfaces with a canonical principal direction.
As before, $\bar{M}^{n+1}$ denotes a Lorentzian manifold endowed with a timelike closed conformal vector field Z, M is an orientable spacelike hypersurface of $\bar{M}$ with a normal timelike unit vector field ξ and $\theta \geq 0$ measures the hyperbolic angle between ξ and Z. By Theorem 2.3, locally $\bar{M}$ is isometric to $-I \times_\varrho N$. In this case, we denote by $h : M \to \mathbb{R}$ the height function of M, i.e., the restriction of the projection $\pi : -I \times_\varrho N \to I$ to M. We may suppose further that locally M is given as the graph of a function $F : U \to I$, where U is an open set of N:

$$M = \{ \ (F(x), x) \mid x \in U \ \}.$$

Note that our definition of a graph use the order of the factors according to the standard use of the notation $-I \times_\varrho N$ for warped products.
The next result is an extension of Theorem 5 in [3].

Theorem 2.8. *Using the notations given above defined as well as those of the previous section, the following statements are equivalent:*

1. *M has a canonical principal direction relative to Z, i.e., T is a principal direction.*
2. *The angle θ between Z and ξ is constant along the directions tangent to M and orthogonal to T.*

In addition, if we consider an open subset of $\bar{M}$ isometric to a warped product $-I \times_\varrho N$ where M is the graph of a function F, the above conditions are equivalent to the following:

3. *The integral curves of T are geodesics in M.*
4. *The norm of the gradient of h is constant along the level curves of h.*
5. *The norm of the gradient of F is constant along the level curves of F.*

Proof. (1) implies (2): If T is a principal direction, $A^\xi(T)$ is a multiple of T, so that (11) implies that $(\tanh\theta)\nabla_T T$ is also a multiple of T; from (8) we have

$$W \cdot \theta = \tanh\theta \langle \nabla_T T, W \rangle = 0$$

for each vector field W tangent to M and orthogonal to T.
(2) implies (1): From the hypothesis and (8), $\tanh\theta \langle \nabla_T T, W \rangle = 0$; but as T is a unit vector field, $\langle \nabla_T T, T \rangle = 0$ and then $\nabla_T T = 0$; substituting in (11) we have that $A^\xi(T)$ is a multiple of T.
From now on we suppose that $\bar{M}$ is given as a Lorentzian warped product $-I \times_\varrho N$, $Z = \varrho \partial_t$ and M is the graph $(F(x), x)$ of a function.
The equivalence of (1) and (3) follows almost the same argument as before; just note that $\tanh\theta \neq 0$ for the case of M being a graph.

A frame field tangent to M is given by

$$E_i = \frac{\partial F}{\partial x_i}\partial_t + e_i, \quad i = 1, \ldots, n.$$

where e_i denotes the lifting to $\bar{M}$ of a n-frame tangent to N. The vector field defined by

$$\xi = (\varrho \circ F)^2 \partial_t + \nabla F$$

is normal to M. Now the height function $h : M \to \mathbb{R}$ is given by $h(F(x), x) = F(x)$. (Incidentally, this expression shows that each level curve of h corresponds exactly to a level curve of F.) Since

$$\langle \nabla h, E_i \rangle = E_i(h) = e_i(F) = \frac{\partial F}{\partial x_i} = \langle \partial_t, E_i \rangle = \left\langle \partial_t^T, E_i \right\rangle,$$

the gradient ∇h of the height function is precisely the component of ∂_t tangent to M. This component can be calculated as

$$\partial_t - \frac{\langle \partial_t, \xi \rangle}{\langle \xi, \xi \rangle}\xi = \frac{1}{\langle \nabla F, \nabla F \rangle - (\varrho \circ F)^2}\left(\langle \nabla F, \nabla F \rangle \partial_t + \nabla F\right).$$

In other words, ∇h and ∇F are related by

$$\nabla h = \frac{1}{\langle \nabla F, \nabla F \rangle - (\varrho \circ F)^2}(\langle \nabla F, \nabla F \rangle \partial_t + \nabla F).$$

Hence, the relation between $\langle \nabla h, \nabla h \rangle$ and $\langle \nabla F, \nabla F \rangle$ is

$$\langle \nabla h, \nabla h \rangle = \frac{\langle \nabla F, \nabla F \rangle}{(\varrho \circ F)^2 - \langle \nabla F, \nabla F \rangle}.$$

Conversely, we may express $\langle \nabla F, \nabla F \rangle$ in terms of $\langle \nabla h, \nabla h \rangle$:

$$\langle \nabla F, \nabla F \rangle = \frac{(\varrho \circ F)^2 \langle \nabla h, \nabla h \rangle}{1 + \langle \nabla h, \nabla h \rangle},$$

Now it is easy to prove that items (4) and (5) are equivalent: Take a level curve of F, which as pointed out before, corresponds precisely to a level curve of h. From the above expressions and the fact that $\varrho \circ F$ is constant along such a curve it is clear that $|\nabla F|$ is constant along the level curves of F iff $|\nabla h|$ is constant along the level curves of h.

To finish the proof, we prove the equivalence between items (3) and (4). From the above we have that $T = \nabla h / |\nabla h|$ and then $\nabla_T T = 0$ is equivalent to

$$\nabla h \left(\frac{1}{|\nabla h|} \right) \nabla h + \frac{1}{|\nabla h|} \nabla_{\nabla h} \nabla h = 0;$$

that is, $\nabla_T T = 0$ if and only if $\nabla_{\nabla h} \nabla h$ is an scalar multiple of ∇h. For every $Y \in \mathfrak{X}(M)$ such that $\langle Y, \nabla h \rangle = 0$ we have

$$Y \langle \nabla h, \nabla h \rangle = 2 \langle \nabla_Y \nabla h, \nabla h \rangle = 2 \langle \nabla_{\nabla h} \nabla h, Y \rangle. \tag{9}$$

Hence, $\nabla_{\nabla h} \nabla h$ is an scalar multiple of ∇h if and only if $Y \langle \nabla h, \nabla h \rangle = 0$ for every Y orthogonal to ∇h, which happens if and only if $|\nabla h|$ is constant along the level curves of h. $\square$

Corollary 2.9. *Let $\bar{M}^{n+1}$ be a Lorentzian manifold endowed with a timelike parallel vector field Z. If M is an orientable spacelike hypersurface of $\bar{M}$ with a canonical principal direction relative to Z then*

$$\begin{aligned} A^{\xi}(T) &= (T \cdot \theta)T + \tanh\theta \nabla_T T, & (10)\\ A^{\xi}(W) &= \tanh\theta \nabla_W T, & (11)\\ \alpha(T, W) &= 0, \quad \alpha(T,T) = -(T\cdot\theta)\xi, & (12)\\ \tanh\theta\langle \nabla_T T, W\rangle &= 0. & (13)\end{aligned}$$

where W is as above, ξ is a timelike unit vector field normal to M and $\theta \geq 0$ measures the hyperbolic angle between ξ and Z.

Proof. We should observe that when Z is parallel, $\varphi = 0$ everywhere. Since, M has a canonical principal direction it follows that $W \cdot \theta = 0$ for every W orthogonal to T. □

Now we will consider the special case when the hyperbolic angle θ of M is a nonzero constant and the vector field Z is parallel. Note that if $\theta \equiv 0$, Remark 2.5 implies that the vector fields ξ and Z are collinear. This in turn implies that M is orthogonal to Z and by Theorem 2.3, M is part of a totally umbilic leaf of the integrable foliation of $Z^{\perp}$.

Corollary 2.10. *Let M be a spacelike hypersurface in $\bar{M}$ with nonzero constant hyperbolic angle function θ relative to a timelike parallel vector field Z on $\bar{M}$. Let W be a section of TM so that T, W are orthogonal. Then*

$$\begin{aligned} A^{\xi}(T) = 0, &\qquad A^{\xi}(W) = \tanh\theta\nabla_W T,\\ \alpha(T,W) = 0, &\qquad \alpha(T,T) = 0,\\ \nabla_T T = 0. & \end{aligned}$$

In particular, T is geodesic in the ambient $\bar{M}$ not only in M.

Proof. Since the hyperbolic angle θ is constant each of its derivatives vanishes. Finally, the equation $\tanh\theta\langle\nabla_T T, W\rangle = 0$ (valid for every W orthogonal to T) of Corollary 2.9 implies that $\nabla_T T = 0$. Here we applied that $\tanh\theta \neq 0$. □

Corollary 2.11. *Let M be a spacelike surface in a three dimensional Lorentzian manifold $\bar{M}^3$. Let us assume that M has constant hyperbolic angle function θ relative to a timelike parallel vector field Z on $\bar{M}$. If M is maximal then it is totally geodesic and if the constant angle is nonzero then it is also flat.*

Proof. First case: If $\theta = 0$, M is orthogonal to Z. Therefore, M is totally geodesic because Z is a parallel vector field.
Second case: Consider $\theta \neq 0$. Since M is maximal, $\alpha(W,W) = -\alpha(T,T) = 0$ by Corollary 2.10. The same Corollary gives $\alpha(T,W) = 0$, and we have that α vanishes identically. This proves that M is totally geodesic.
To prove the second part, we will see that $\nabla_W W$=0. Since $|W| = 1$, we have $\langle\nabla_W W, W\rangle = 0$. On the other hand, using that T and W are orthonormal, Corollary 2.10 and the condition of maximality, we have

$$\begin{aligned}\langle\nabla_W W, T\rangle &= -\langle W, \nabla_W T\rangle = -(1/\tanh\theta)\langle W, A^{\xi}(W)\rangle\\ &= -(1/\tanh\theta)\langle\alpha(W,W),\xi\rangle = (1/\tanh\theta)\langle\alpha(T,T),\xi\rangle = 0.\end{aligned}$$

So, $\nabla_W W$=0. This fact and the equality $\nabla_T T = 0$ in Corollary 2.10 prove that M has constant zero curvature. □

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