On the analytic and synthetic demonstrations in Fermat's work on the law of refraction

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Abstract

The history of refraction is extensive and quite complex, it can be dated back to Ptolemy and Ibn Sahl, and the law of refraction itself is usually called Snell's law. We do not wish however, to give a complete account of this history, our aim in this paper is rather to study Fermat's contributions which are divided into two periods. In the first period, commencing with the publication of Descartes' *Discours de la Méthode*, Fermat rejects the law of refraction as proposed by Descartes claiming that the method used to obtain it is flawed and that the law must therefore be wrong. During the second period, dating from 1657 to 1662, Fermat presents two proofs of the law, one which is analytic and one that is synthetic, and concludes (to his own surprise) that Descartes' law was in fact correct. Our goal is to study in detail in what ways these proofs are analytic and synthetic and then account for Fermat's results given he had set out to disprove them.

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Introduction

Our aim in this article is to achieve a better understanding of the two proofs proposed by Pierre de Fermat of the problem of refraction of a ray of light passing from one medium to another. The two proofs appeared in texts named *Analysis ad refractionem* and *Synthesis ad refractionem* respectively and we wish to study why and in which sense one of them is presented as *analytic* and the other as *synthetic*. To our knowledge this has not been done. There is a vast amount of literature surrounding Fermat's career and a vast amount of literature regarding the problem of refraction,[1] however a profound study of what makes the proof in *Analysis ad refractionem* analytic and what makes the proof in *Synthesis ad refractionem* synthetic is not available and this is our intention here. To achieve our goal, it will also be necessary to study the problem of refraction as it is treated in Fermat's correspondence and to analyse the context of Fermat's intervention in the vast and long debate concerning the law of refraction that took place during the seventeenth century.

Fermat's interventions are clearly located in two short, but widely spaced, periods of time, with the second period itself being split into two moments. The first period covers the years 1637 and 1638 in which the starting point can be pinpointed as the publication of the Descartes' *Dioptrique* in 1637. During this first period several letters are exchanged by Fermat and Mersenne, concerning Descartes' *Dioptrique*, but also concerning Fermat's objections to Descartes' law. The second period extends from 1657 to 1664 and can be divided into two: the first period dating from 1657 to 1658 and the second one from 1662 with a final letter to an unknown addressee in 1664; the authors concerned in this second period are Fermat, Cureau de la Chambre, Clerselier and the anonymous author denoted by M de ****.

In addition to being two clearly distinct chronological periods, the first period can be mainly characterized as Fermat's reaction towards Descartes' *Dioptrique*, where it is only his objections, be them correct or not, that are discussed. That is, during this first period Fermat does not propose an alternative idea to explain refraction. The second period starts in 1657 with a letter from Fermat to Cureau de la Chambre in which Fermat proposes for the first time an original explanation of the phenomenon of refraction of light. This idea was not immediately discussed by the Descartes' followers, Clerselier in particular, who was certainly aware of the letter. In 1658, the debate between Fermat and Clerselier focusses around Descartes' explanation in the *Dioptrique*.

It was only four years later, in 1662, in a letter to de la Chambre, that Fermat returned to his own explanation of refraction, based on his old method of maxima and minima, in his text on the *Analysis ad Refrationem*. This second letter to de la Chambre opens the last debate regarding refraction: Clerselier has now read Fermat's letter and criticises his

1 We refer the interested reader to Andersen 1983, Giusti 2009, Mahoney 1973 and Sabra 1981.

explanation of refraction, even though Fermat had clearly expressed to de la Chambre that his explanation had in fact, through a different method, confirmed Descartes' law.

By 1664 it becomes clear that even though Fermat changed his mind, being first convinced that Descartes' law should be false since it was obtained from incorrect arguments, and at the end finding the same law, he does not recognize the validity of Descartes' arguments. On the other hand, Clerselier, representing the Cartesians, seems to never have taken Fermat's arguments seriously, and not with much enthusiasm accepted Fermat's confession that, though his criticisms towards Descartes' law of refraction were based on his distrust of his method of proof, he finally recognized that Descartes was right.

We believe it is important to give a brief account that reflects the general context as it relates to the issue under discussion to be able to achieve our goal and we will do so in the next section.

Context surrounding the law of refraction

The nature of light holds a weak place in the controversies surrounding optics during this period. There are various reasons for this. The first is that the theories about the nature of light were practically non-existent, weak or otherwise kept out of the debate. We believe that Kepler, Galileo, Fermat, Pascal, Roberval, Mersenne had no stable and thorough doctrine on the nature of light. Let us add that Descartes, who did have such a doctrine, did not use or present it in his *Dioptrique*. The presentations of other inventors of theories on the nature of light (for instance, Huygens, Newton and even Leibniz) date from later on. A reference text on this *ignoramus* –the nature of light– is Pascal's letter to *Père Noël* dated on October 29, 1647:

> « Puisque la nature de la lumière est inconnue, à vous et à moi ; que, de tous ceux qui ont essayé de la définir, pas un n'a satisfait aucun de ceux qui cherchent les vérités palpables, et qu'elle nous demeurera peut-être éternellement inconnue, je crois que cet argument demeurera longtemps sans recevoir la force qui lui est nécessaire pour devenir convaincant. Car considérez, je vous prie, comment il est possible de conclure infailliblement que la nature de la lumière est telle qu'elle ne peut subsister dans le vide, lorsque l'on ignore la nature de la lumière. Que si nous la connaissions aussi parfaitement que nous l'ignorons, nous connaîtrions peut-être qu'elle subsisterait dans le vide avec plus d'éclat que dans aucun medium [...] Ainsi, remettons cette preuve au temps où nous aurons l'intelligence de la nature de la lumière ». (Pascal 1964-1992, 2: 520)

This point is crucial in what follows because it shows one of the main issues of the debate between those involved. In other words, the nature of light will not be allowed as the point of departure in proofs about the law of refraction.

Since the beginning of the century, the problem of refraction took an increasingly important role in the discussions within scientific circles. The reasons for this are quite diverse: the use of telescopes that in turn led to the development of astronomic optics,[2] important works on optics such as those of Kepler which followed those of Della Porta, de Grimaldi, de Francesco Sizi, Nicéron, Snellius, Mersenne, and Cavalieri (1632). However, the most important event, and certainly so, in the eyes of Fermat was the publication of Descartes' *Dioptrique*.

Another reason, which is very general in nature, contributed to the understanding of the importance of the question of the *path of light*; we could name it the epistemological situation. This is in fact a very old problem, considered difficult or impenetrable by Binet, Pascal and Galileo. The phenomenon of refraction (and its colourful effects as they appear in the rainbow) seemed to set the limits of what could be discovered by natural science. The methods of physics would then be tested by this problem: would it be overcome by means of efficient causes, final causes, by mathematics (which could then play the role of formal causes) or by empirical methods? Can the science of the moderns, as they called themselves, overcome this obstacle? And if so, by what route?

La Dioptrique plays the role of a catalyst, presenting a general solution and, at the same time, opening a controversy. The law Descartes sets out seems correct except in the eyes of Fermat; we will return to this point further on in our text. The method caused enthusiasm in some, a test of refutation in others. Many other proofs of the result flourished after this publication. These proofs are similar in that they validate the minimal version of the law, that is, the sines are in a constant relation; on the other hand, the principles and methods used to establish it offer a variety rarely achieved in science.

Fermat's intervention on this topic ends in 1664 and we note that the three great theoretical texts concerning light in general, and refraction in particular, did not appear until the following decade: Newton writes his *Opticks* in 1672, Leibniz his *Unicum principium* in 1672 and Huygens his *Traité de la lumière* in 1678.

Descartes' account of refraction

In 1637, in the Discours I of his *Dioptrique*, Descartes assumed that refraction, as an optical phenomenon, could be explained using the image of a moving ball « Car il est bien aysé

2 It is clear that the telescope is connected with the refraction law and, according to this, Descartes writes, in the first page of the *Dioptrique*: « Mais, à la honte de nos sciences cette invention si utile et si admirable, n'a premièrement été trouvée que par l'expérience et la fortune» (Descartes 1897-1913, 6: 81-82).

à croire que l'action ou l'inclination à se mouvoir ...doit suivre en cecy les mesmes loys que le mouvement » (Descartes 1897-1913, 6: 89). Even if the deviation of a light ray is imagined as a deviation in the path of a moving body, Descartes looked for no other resemblance, so that this implicit assumption of a corpuscular nature of light plays no further role in his explanation of refraction. His explanation, given in the Discours II, rests upon a distinction between the movement of the ball and the *determinatio* of this movement, and a diagram designed to illustrate not only this difference but also the assumptions and consequences obtained from them.

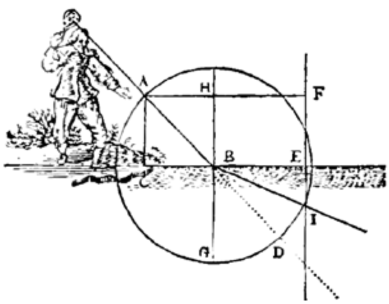

Figure 1

« ...supposons qu'une bale, poussée d'A vers B rencontre au point B [...] une toile CBE [...] en perdant seulement une partie de sa vitesse, à scavoir, par exemple la moitié. Or cela posé, afin de scavoir quel chemin elle doit suivre, considerons de rechef que son mouvement differe entierement de sa determinatio a se mouvoir plustost vers un costé que vers un autre, d'où il suit que leur quantité doit estre examinée separément. Et considerons aussy que, des deux parties, dont on peut imaginer que cete determinatio est composée, il n'y a que celle qui faissoit tendre la bale de haute en bas, qui puisse estre changée en quelque facon ... et que pour celle qui la faisoit tendre vers la main droite, elle doit tousjours demeurer la mesme qu'elle a esté... ». (Descartes 1897-1913, 6: 97)

The line AB (see Fig. 1) is nothing but the image of the path of the ball in the air, a movement taken in a certain time t_0, and at this moment the immediate identification of the line AB with the *determinatio* of the movement appears. The *determinatio* of the movement itself has two *components* or parts, whose quantities ought to be considered and examined separately; they are represented by the lines AH and AC, the first one representing the component of this *determinatio* of the movement towards the right, while it is the second line AC which represents the vertical or falling *determinatio*. In each case the length of the line represents the quantity of this *determinatio*. Now, Descartes presents the hypothesis that only the vertical component of the *determinatio*, or the vertical *determinatio*, changes (its quantity is modified) while the horizontal one does not change, because only the first one is modified when the ball passes through the line CBE which divides the two media. A change in the velocity is assumed, from an initial constant velocity v_0 in the first medium to a second velocity in the second medium v_f which is taken to be one half of v_0: $v_f = \frac{1}{2} v_0$. This explicit assumption becomes, when combined with the explicit claim that the horizontal *determinatio* does not change at all, the only justification for a geometric conclusion imposed to the diagram: the line FEI being such that BE = 2BC, becomes the quantity of the horizontal *determinatio* in the second medium since it has not changed but it has now been traversed in double the time, that is $2t_0$, hence it has traversed a horizontal distance twice as long.[3] There is certainly no explanation of why an assumption over the vertical *determinatio* could not provide a correct answer; that is, considering under the line CBE a distance BK taken to be half of HB, and claiming that, under the same assumption that $v_f = \frac{1}{2} v_0$ since the vertical *determinatio* is the one which has been modified, the distance traversed in this vertical direction/*determinatio* is necessarily half of the first one. Hence if the parallel horizontal line KI' was drawn (taking I' on the circumference), the line BI' would be the line describing the path of the ball in the second medium or, the refracted light ray. In this case the law of refraction would become not a sine law but a false or incorrect cosine law, and Fermat is certainly right when he claims that no justification has been provided by Descartes. The sine law appears as a consequence of a conservation principle which has not been, and could not have been, stated as such. In fact, we know that the sine law is equivalent to the cosecant law, where the sine ratio $\frac{AH}{KI}$ is replaced by the cosecant ratio obtained from unequal circles, that is, the ratio of the radii of these circles is the ratio of the change of velocity when the ball passes from one medium to the other. In other words, there are only two possible, and equivalent, ways to find the correct solution: the ratio of the two velocities is given in a unique circle as a ratio of two different components or values of the horizontal *determinatio*, or it is directly given as the ratio of two radii of circles, where these radii describe the complete *determinatio* (and quantity) of the movement, confirming in this way that it is the conservation principle is the key assumption.

Descartes provides an explanation where no confusion is made between *determinatio* of the movement and the force or velocity of this movement, his claim is that, though being different, a variation in one of them produces a change in the other, a variation that ought to be adjusted and fixed between them, since this is exactly what produces refraction as an optical phenomenon in nature. Of course, since the mechanical image of a moving ball has been proposed as the key image for the explanation of the refraction of a ray of light, besides the implicit assumption of the corpuscular nature of light, a second hypothesis (consistent with the first one) assumes that the moving ball actually represented the movement of a light corpuscle, or else the ease or difficulty of light to penetrate a dense or a rare medium becomes immediately an increase or decrease of the velocity of this light ray. In other words, the refraction of a ray of light, as explained by Descartes in his *Cogitationes Privatæ* (Descartes 1897-1913, 10: 213ff), being a deviation produced by the ease or difficulty to be generated, according to the nature (dense or rare) of the medium from which it comes, to the nature of the medium towards which it continues, becomes

3 The fact that the distance BE is equal to 2CB could also be interpreted as follows: before the impact the ball moves along the line AB during a time t_0 but since after the impact its velocity becomes $v_f = v_0/2$, the ball moves along the line BI in a time $t_f = 2t_0$ hence, since the horizontal *determinatio* does not change at all, it has now this double time to act or cause the horizontal movement of the ball, that is why in the horizontal direction its position is given by the line FEI whose distance from the line HB is now double the distance CB.

in the *Dioptrique* a consequence of the variation in the velocity of the movement when a moving body passes from one medium to another.

But in addition to the pertinence or validity of the mechanical image of the moving ball to provide a clear explanation of the refraction of a ray, several magnitudes, despite their possible correlation, ought to be represented in Descartes' diagrammatic image. These include the angles of the respective rays (incidence and refracted) with respect to the normal of the refracting surface, and the *resistance* or *difficulty* of a light ray to go through a medium. This last magnitude ought to be quantified, as the *degree of resistance* (or *difficulty*) of transmission of a ray of light which is proportional to the *degree of density* of a medium (directly proportional according to Snell and Fermat, inversely proportional according to Descartes). But Descartes' conclusion in the *Dioptrique* states that since the degree of penetration/generation of light is proportional to the density of the medium, the increase/decrease of this penetration/generation becomes an increase/decrease of the normal component which, in his own words, becomes the (vertical) *determinatio* and which leads to a refraction towards/away from the normal. Of course, this way of reasoning leads to a cosine law as we mentioned above, unless this principle is accompanied by another conservation principle as we have already pointed out.

Our aim here, however, is not to give a detailed description of Descartes' ideas and their evolution from the *Cogitationes* to the *Dioptrique*, but rather to understand why, given this confusion, Fermat remains sceptic and rejects the mechanical argument, without regard to the adjustments introduced so that this image works correctly.[4] No other argument supports Fermat's own proof, and even if we consider that his principle sounds as one proposed in the same framework of natural philosophy, where the main goal is to understand how nature behaves, he will find the way to provide an *analytic* proof from this argument via his maxima and minima method. We will study this in detail in a later section, but first we believe it is relevant to call attention to the fundamental disagreements between Descartes and Fermat.

4 As it is well known, a turning point in the evolution of the ideas of Descartes is his collaboration with Beeckman and Mydorge; in fact, it is in his letter to Beeckman in 1628 (dated by AT) where we find the first recorded statement of the sine law of refraction. However, some scholars claim that another turning point is the moment when his "corpuscular-mechanical" theory of light is proposed, in his treatise *Le Monde*, which is also called a treatise on light.

Fermat and Descartes: three key divergences

From the beginning of the exchange to the end, hostility to Descartes is a constant feature of Fermat's letters. The formulas of civility hardly conceal the depth and breadth of his anti-Cartesianism. When he must finally yield, the surrender is accompanied by a denigration that devalues the victory of the opponent. In substance, here is what he concedes to Cartesian success: it resembles the victory of a conqueror whose fame is such that he carries a city without having to fight, and we know the very relative value of fame. We will not try to discover the origins of this hostility. We will limit ourselves to see in what domains and in what directions it manifests itself.

There are three persistent criticisms made by Fermat to Descartes. A first one, that he will never give up in the 25 years of intermittent but solid dispute, is that according to Descartes, the movement of light is easier (although Fermat sometimes says faster) in a dense medium than in a rarer one. For Fermat, this goes against common sense and is totally unacceptable. On several occasions Fermat recalls that he is not a physicist; yet he is absolutely certain of the truth of the opposite thesis, which is nevertheless of a physical nature, according to which light moves more rapidly (or more easily) in air than in water. It is interesting to question where this certainty comes from and it goes without saying that Descartes' point of view seems incomprehensible to him. Yet, on this issue, opinions are divided and rely on further knowledge to decide. Fermat himself seems to be evolving on the related question of the instantaneous movement of light. Take for instance the letter he wrote to Mersenne in September 1637:

> « La lumière pénètre en un instant les corps diaphanes et semble n'avoir rien de successif. Mais la géométrie ne se mêle point d'approfondir les matières de physique ». (Fermat 1891-1912, 2: 109)

The second criticism is of a philosophical nature. For Fermat, it seems that the deployment of a finalist principle of minima is out of possible dispute as well as perfectly natural, while Descartes, in the *Dioptrique*,[5] does not use it. And in fact, Clerselier will adamantly argue, especially in his letters to Fermat (1658-1662), that the use of such principles in physics is incongruent in itself. This is enough to confirm the preceding remark about Fermat's statements claiming that he wasn't a physicist; he has just admitted to making an incursion into this science (to both C. de la Chambre and Clerselier). He is, however, very strongminded as to the methods that are suitable for this science whose development occurs by the association of finalist principles and mathematics. On this subject –the methodology of the physical theory– Fermat's point of view on the role of experimental data will be em-

5 This is also true of his other physics treatises.

phasized. He defends the rather pertinent position that experiences and measures cannot settle the debate. He is aware of the work of various experimenters (including P. Petit), which complies with Descartes' sine law. Fermat insists on suggesting that other laws and other proportions may be suitable for measurements, given the precision of these. It is important to note that, even until the end of 1658, Fermat believed that the Cartesian law would have to be invalidated, even though experimental results supported it. In March 1658, he maintained this position in a letter to Clerselier: « je prétends que la véritable raison ou proportion des réfractions est encore inconnue » (Fermat 1891-1912, 2: 374).

It is worthwhile to note, that this second disagreement with Descartes appears only in the second period (from 1658 onwards) after Fermat formulates his own explanation of refraction, while the first, as well as the third, endure throughout the two periods described, that is, throughout 25 years.

Finally, the third issue of an anti-Cartesian criticism, which is essential, is in fact twofold. Descartes has no basis in using an analogy to deal with a question of physics, nor is he justified in using the decomposition of motion as he does. These two issues are difficult, and it is interesting to specify the controversy over these two choices of the Cartesian proof. In this section, we present the general characteristics of these Cartesian decisions; and in the next section, we examine with more detail Fermat's disagreements.

The first choice is clearly epistemological. Descartes does not make use of a direct analogy; he does not maintain that light is like a ball; he sustains that a common model can make the movement of both comprehensible. To have a correct evaluation of the Cartesian way, it is necessary to have in mind the advertisement given at the beginning of the *Dioptrique*:

> « Or, n'ayant ici autre occasion de parler de la lumière, que pour expliquer comment ses rayons entrent dans l'œil, et comment ils peuvent être détournés par les divers corps qu'ils rencontrent, il n'est pas besoin que j'entreprenne de dire au vrai quelle est sa nature, et je crois qu'il suffira que je me serve de deux ou trois comparaisons, qui aident à la concevoir en la façon qui me semble la plus commode, pour expliquer toutes celles de ses propriétés que l'expérience nous fait connaître, et pour déduire ensuite toutes les autres qui ne peuvent pas si aisément être remarquées; imitant en ceci les astronomes, qui, bien que leurs suppositions soient presque toutes fausses ou incertaines, toutefois, à cause qu'elles se rapportent à diverses observations qu'ils ont faites, ne laissent pas d'en tirer plusieurs conséquences très vraies et très assurées ». (Descartes 1897-1913, 6: 83)

One can easily see that he maintains that the movement of light is, in some respects, like that of the ball. He explicitly writes that analogies allow us to not find strange some properties of light : « Ce qui vous empeschera d'abord de trouver estrange, que ceste lumiere puisse estendre ses rayons en un instant [...] Vous ne trouveres pas estrange nos plus, que

[…] » (Descartes 1897-1913, 6: 84). Thus, analogies are pillars for the imagination. Descartes studies a property of light and, to do so, he proposes tools that will help understand what is happening (he does not pretend that this is actually happening). These tools are geometric properties that are relevant for material movements in general (like that of a ball). Descartes justifies this method by referring to astronomers who use false assumptions to understand phenomena. These *false assumptions* are geometrical ones (like eccentric circles for instance).

We are dealing with one of the first modern cases of the use of a mathematical model within a physical theory. We find here an interesting orientation of Cartesian epistemology which frequently escapes the presentations that are made on this subject.[6] This method is coupled with the use of mathematical expressions (geometric in this case) of the quantities and processes that characterize the phenomenon that is studied. In other words, during the elaboration of his theory of refraction, Descartes, at some stage of his work, entrusts the discursiveness of reasoning to magnitudes and mathematical operations. These magnitudes and operations have been chosen or invented by him in an arbitrary way, that is, without having to be of a physical nature. It is the result of these operations that will have meaning and physical interpretation, otherwise it would have been inefficient and therefore an invalid choice. Fermat contests this possibility which, in effect, creates an indirect analogy between light and a ball as the same choice of mathematical tools is made to deal with a common aspect of both, namely their motion.

At the heart of these geometric tools are the characteristics of what Descartes calls *determinatio* and this is what Fermat criticises. Knowing exactly what this *determinatio* is, is a difficult point. It is not the movement itself, nor the intensity of velocity, nor the direction, but it is somehow related to the three of them. We could possibly consider it as a proto vector, a new kind of magnitude that has an intensity and a direction, the latter of which can be decomposed according to the parallelogram law. Then it is possible for all three magnitudes, the intensity and the two components, to change. They are distinct but not independent: a change in one of them has consequences on the others. The problem is to decide how these changes, when viewed as geometrical objects, should be imagined.

It is interesting to analyse what the Cartesian result actually is. Descartes claims that the component of the *determinatio*, parallel to the surface that separates the media is not affected when its size (the intensity or velocity) changes (see Section 4 below).[7] If we interpret this in

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6 See G. Federici Vescovini's text, "Descartes et les sciences curieuses : le raisonnement ex suppositione et le Moyen Âge", in Biard and Rashed 1998.

7 It is astounding to consider the way Newton establishes the refraction law in a combination of a proposition of the *Principia Mathematica* (Book I, sect. XIV, prop. XCIV, theorem XLVIII) with propositions of the *Opticks*, in particular, Axiom V from Book I. It is the same as the Cartesian one, considering the rules for the parallel and perpendicular components. To

modern terms, we can translate Descartes' claims into the following: In the first medium, a "vector" V expresses the movement of light; but when the light ray meets the second medium, V is modified in two ways: $||V||$ becomes $k||V||$ where k is a constant parameter characteristic of the two media and if V_h and V_v denote the parallel and perpendicular components of the vector, then V_h is not affected, and then, with respect to the parallelogram law, it is enough to determine V_v. These "rules" are then sufficient to establish the Cartesian proportion of sines.

We have said that Descartes chooses these magnitudes and their properties in an arbitrary way; but the question of how he chooses them must still be addressed. It is certainly remarkable that he has conceived a magnitude having both intensity and direction, and it suits the situation so well that this decision is immediately convincing. What is more difficult to accept is the choice of rules for the modification of the components. However, it is inevitable to accept that the two components are modified differently when moving from one medium to another; indeed, if it were not the case, the ray of light would not be deviated. It is therefore rational to suggest that while one of the components does not change, the other does; and the question of how it changes needs to be studied. It is basically very simple if two things are specified: firstly, the ease of displacement (or speed, if one prefers) is modified in a characteristic proportion of the two media, and it is the size of the *determinatio* that is affected; and secondly, the parallelogram law is valid to decompose and recompose the *determinatio*-vector.

Fermat will never accept this as the underlying question and so the question whether these changes are truly physical in nature remains. This question does not make much sense in the Cartesian dioptric: their role is to provide a geometrical model whose result is capable –or not– of correctly accounting for the phenomenon of refraction and this is what happens. It is a mathematical model placed at the heart of a scientific theory and this does not fit in Fermat's vision.

Descartes' *Dioptrique* is one of the birthplaces of the concept of vector and it is certainly not a coincidence if it is in a physics text that this mathematical object is born. This will be confirmed by its other inventors. It is clear that the Cartesian tool is not fully developed and that its concept appears confused. The association between the directional characteristic on the one hand and the norm characteristic on the other, is not clearly formalized. The consequence that this criticism has for Fermat is radical, he does not admit the use of the model, and he does not admit the transformations undergone by the invented mathematical object, and his logical conclusion is that there is no Cartesian proof of the law of refraction. Fermat does make use of the notion of *determinatio*, but it is different from that of Descartes. For him, the distinction between movement and *determinatio* is the difference between trajectory and the *tendency to move* in one point.

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this effect, see Jullien 2006.

> « Je doute premièrement, et avec raison, ce me semble, si l'inclinaison au mouvement doit suivre les lois du mouvement même, puisqu'il y a autant de différence de l'un à l'autre que de la puissance à l'acte ». (Fermat 1891-1912, 2: 108-109)

In other words, for Fermat, *determinatio* is a notion of physics, distinct from the other notion of physics which is the movement accomplished by a mobile. For Descartes, in fact, it is quite another thing, the *determinatio* is a geometrical object, a magnitude. It is a useful mathematical tool for understanding movement.

Fermat's objections to Descartes' law of refraction as seen in his correspondence

As we have said, Fermat's dealing with the law of refraction occurs in two distinct periods of time and his analytic and synthetic demonstrations of the law occur at the end of the second period after a long discussion with some of Descartes' followers. Throughout this discussion, Descartes' followers present (and defend) Descartes' principle exactly as it was explained in his first essay of his *Discours de la Méthode*, the *Dioptrique*.

While in November 1637 Fermat explains to Mersenne his doubts about Descartes' law of refraction, regarding not only his conclusion and the proportion that he found, but also, and mainly, the way in which this law was actually found; at the end of this long exchange of letters with Cureau de la Chambre and Clerselier, in 1662, we see Fermat explaining that he found the same proportion as Descartes, but through a completely different method.

It is not difficult to see that Fermat's method in 1662 is different from Descartes', a difference that had been made manifest after the publication of the *Discours*. Fermat claims that the correct method for proving the law of refraction is twofold: first is the use of a principle, qualified as metaphysical, or even moral, by the Cartesians, claiming that nature always acts by searching the shortest ways and means, then the use of the method of maxima and minima. It is rather remarkable the way Fermat claims, and this plays a key role in the characterisation of his proof as *analytical*, that his method of maxima and minima, which is by this time more than 20 years old, provides the correct geometrical expression of this natural principle.[8]

8 We do not know with certainty when Fermat's method of maxima and minima was discovered, but we do know that his treatises *Methodus ad Disquirendam Maximam et Minimam* and *De Tangentibus Linearum Curvarum*, were written in a short period of time in 1637 and sent to Mersenne in December 1637. In a letter to Roberval dated on September 22, 1636 Fermat claims that he had communicated his method to Despargnet as early as 1629. For a detailed account see Strømholm 1968.

In a letter sent to Cureau de la Chambre in August 1657, which we take as opening the second period, Fermat insists on his objections towards Descartes' method of decomposition of movement, and for the first time he proposes an explanation of his own, claiming that the correct method to provide the explanation and justification (*la raison*) for refraction should be based upon the physical principle already mentioned, though translated into a general geometric problem and solved by his method of maxima and minima.

It is noteworthy that after this letter to de la Chambre, not a single word about his new and correct method is mentioned in a series of letters that he exchanged with Clerselier, the first one written in March 1658, to yet again reject Descartes' method and proof. It is in another letter to de la Chambre in January 1662 that his claim of agreement and distance with Descartes is clearly stated: Descartes' law is correct; Descartes' method of proof is certainly not appropriate.

In what follows we will present a comprehensive view of these letters and of Fermat's arguments to give a correct analysis of the contextual background of his analytic and synthetic proofs. Fermat's first claim, as expressed to Mersenne in 1637, is that Descartes' proof is not correct as he has made several mistakes which must be pointed out, to mend his (clearly) false conclusion. The first of Descartes' mistakes is providing an explanation of the deviation of a ray of light using a *mechanical* image. However, this difficulty does not appear concerning the *determinatio* of the light ray; moreover, the refraction of a light ray is to be explained as a change or variation within the *determinatio* of this ray. We could say, and this is Descartes' explanation in his *Dioptrique*, that this decomposition has been proposed using the parallelogram law, however Fermat is right since no proof has been provided for this law, and no proof would be provided for the rest of the 17th century (or maybe even later). Once again, whether a proof is proposed, or it is assumed as an axiom, the parallelogram law appears in (almost) every mechanical explanation, but it is not clear, according to Fermat's requisite, that it is appropriate for the explanation of refraction.

With these objections in mind, Fermat sent a letter to Mersenne a few weeks after his first reading of Descartes' *Discours*. In this letter, we can find a clear objection towards the *Dioptrique* and the *refraction law* which is supposed to be proved therein. The same objections towards the mechanical explanation and the image of a moving ball are raised 20 years later in the first letter sent to Clerselier on this subject (in March 1658). However, his objections towards the law of refraction itself are not explicitly raised in this letter, in fact a light sympathy is expressed. The beginning of this letter is worth quoting in detail:

> « Les conclusions qui se peuvent tirer de la proposition qui sert de fondement à la Dioptrique de M. Descartes sont si belles et doivent naturellement produire de si beaux effets dans tous les ouvrages de l'art qui regardent la réfraction, qu'il seroit à souhaiter, non seulement pour la gloire de notre défunt ami, mais bien plus pour l'augmentation et embellisement des sciences, que cette proposition

fût véritable et qu'elle eût été légitimement démontrée, et d'autant plus qu'elle est de celles dont on peut dire que multa sunt falsa probabiliora veris (many probable truths are false) ». (Fermat 1891-1912, 2: 367)

Here we have Fermat explicitly stating that he would like to be wrong, to be convinced that his previous objections towards Descartes' law of refraction appear to be true, and his objections to be absurd: « je me mets donc, Monsieur, dans la posture d'un homme qui veut être vaincu, je le declare hautement » (Fermat 1891-1912, 2: 368). Unfortunately, the objections he raised against Descartes' claim and proof have not been banished by Descartes' followers.

Descartes' assumptions are made once he obtained the equality of the angles in the case of reflection, however in the case of refraction the main objection raised 20 years before remains the same in Fermat's opinion, and they come from the distinction (as Fermat conceives it, which is very different than how Descartes views it) between the movement itself and the *determinatio* of this movement, a distinction that is assumed but is not clearly followed. There are in fact two assumptions made by Descartes and which are not clearly stated in his explanation of refraction: the component of the *determinatio* of the ball from left to right suffers no modification, but the component of the *determinatio* from top to bottom does. This modification is explained as a consequence of the change in the size of the *determinatio*. Hence the distinction between *determinatio* and movement (with its velocity and force) appear to be mixed: the change in the *determinatio* of the movement from top to bottom is explained as a consequence of a change in the force and/or velocity. The same applies for the other component of the *determinatio*, the horizontal component (from left to right) is assumed to remain without any change, while the global change in the *determinatio* of the ball is explained as a consequence of the fact that this horizontal *determinatio* remains the same, and also, as a consequence of the fact that it is the velocity which remains the same, therefore the variation or change in the *determinatio* of the ball, which is exactly what refraction is, appears as a consequence of a modification in the velocity from top to bottom and the conservation, without change, of the velocity from left to right. The independence of these two notions, movement and *determinatio* of movement disappear, and in fact, it becomes clear that there never was such independence.

These objections remain without answer and complete justification by Descartes' followers, and in a sarcastic way Fermat claims that if a real proof had been provided by the author of the *Dioptrique*, or by any of his followers, he would have obviously already accepted and endorsed this refraction law, since he already acknowledged his longing to be convinced of Descartes' law through a real proof but not by means of a tricky argument.

Here we face a very singular and passionate discussion from a historical point of view, since in addition to all the arguments in favour or against Descartes' proof of the law of refraction given by Clerselier and Fermat, two questions remain: first the possible explanation of the *refraction* of a light ray using an argument involving the movement of a body (which Descartes uses as a *model*), second and closely related to the first, the distinction that exists, between the movement of a body and the direction of this movement and the way in which it should be understood. Concerning this second point it is a remarkable fact that Clerselier and Fermat blame each other for having confused these two notions, a confusion that according to Fermat explains the nature of the (false) proof given by Descartes, and a confusion which, for Clerselier, explains Fermat's misunderstanding of this proof. However, it is very important to note that both in Descartes' *Dioptrique*, in the *Discours II* (Descartes 1897-1913, 6: 97-98) and Clerselier's letter dated May 15th, 1658 (Fermat 1891-1912,2: 383-386) a distinction between *determinatio* and speed is clearly established, but not so with their independence. The change in speed (of the norm, we should say) determines the modification of the components of the *determinatio* (or of the second, if the first is unchanged). In this manner, we could conclude that it is Fermat who does not see the interest of the concept (which is admittedly still not crystal clear) that Descartes is inventing and we add that Newton will see it in the same way.[9] Of course, it is quite confusing as any new concept is, and there is obviously no formalism involved yet, but at the end of the day, it is the key to Descartes' argument.

This debate is as extraordinary as it is since, from our perspective, the misunderstanding originates from both sides. This misunderstanding can be explained and clarified as a consequence of the intrinsic confusion of this distinction proposed by Descartes; there is an explicit claim that these two notions, movement and *determinatio* of movement, ought to be distinguished, but no clear explanation (or definition) is provided for each of them. We must first underline that on several occasions Descartes and Clerselier claim, when trying to clarify this difference, that the movement of the body comes together with its velocity and the force increasing this velocity, while the *determinatio* of the movement, different from the movement itself, would be independent of the velocity and force. We believe this is a fair point as the modification in speed can be used to describe the change in *determinatio*.

This claim is what Clerselier tries to explain to Fermat in May 1658, when he argues, once again, that if in Figure 1 a line CBE divides two different media, say air and canvas or cloth, the *determinatio* of the movement having been divided in two *determinatio*s, a vertical one (from top to bottom) and a horizontal one (from left to right), it appears that the horizontal *determinatio* suffers no change at all and only the vertical one does suffer a modification. After Fermat's objections that this conclusion assumes that the variation of the vertical *determinatio* appears as a consequence of the reduction of the velocity, let's leave apart the proportion in which the velocity decreases, in his movement from top to bottom, showing in this case that there is no such independence of the *determinatio* of

9 See Jullien 2006, 89-95 for comments on this point and in particular regarding Prop. XCIV, Th. XLIII, Prop XCV and Th. XLIX in Newton's *Principia Mathematica*.

the movement with respect to the velocity of the body, Clerselier insists that Fermat's misunderstanding comes from the fact that he has not paid attention to Descartes' claim that the (vertical) *determinatio* of the body just suffers *some kind of change* (« elle peut être change en quelque façon» (Fermat 1891-1912, 2: 384)). It is inacceptable then to admit that there is no change at all, as the claim that this *determinatio* changes completely is also false. In fact, what changes is the *quantity of this determinatio*:

> «Car en effet elle n'est pas tout-à-fait change, puisque la balle continue de descendre, mais elle est change en quelque façon en tant que c'est changer en quelque façon la determinatio qu'un mobile avoit à avancer vers un certain côté, que de faire que dans le même temps il n'avance pas tant vers ce côté-là qu'il faisoit auparavant : ce qui change la quantité de sa détermination». (Fermat 1891-1912, 2: 384)

This explanation could only be understood as a claim that a change in the velocity happens in the vertical *determinatio*, and it is this change which is understood as a change in the quantity of *determinatio*:

> «[...] quoique la balle continue de descender et même qu'elle descende das la même ligne quand elle a été poussée perpendiculairement, on ne sauroit pas dire que cette détermination vers le bas soit la même, mais elle est changée en quelque façon, ainsi que dit M. Descartes. Car la balle ne descend plus avec la même quantité de détermination, puisque dans un temps égal elle ne va pas si loin qu'elle étoit déterminée d'aller Avant qu'elle eût perdu la moitié de sa vitesse, ce qui est un changement en la détermination qu'elle avoit à avancer vers ce côté-là». (Fermat 1891-1912, 2: 388)

However, Clerselier insists that while this change happens in the vertical *determinatio*, the horizontal one suffers no change at all, even if a change of velocity occurs, since «[...] cette détermination peut demeurer la même quoique la vitesse soit changée, une même détermination pouvant être jointe à différentes vitesses» (Fermat 1891-1912, 2: 384). Hence to repeat Descartes' claim in a different context, as quoted by Clerselier, we could say that the movement (together with its velocity and force) is different and should not be confused with the *determinatio* of the movement; nevertheless, *some kind of relation* exists between them, since a variation of a velocity does not modify the *determinatio* of the movement, even if in the vertical *determinatio* the variation in its quantity comes from a variation of this (vertical) velocity.[10]

10 In modern terms, it is clear that if a vector's norm changes and one of its components does not, then the change in the other component is determined. This is what was being pointed out, the fact that the norm is truly distinct form the components but not independent of them.

Fermat has clearly claimed that the *determinatio* of the movement is to be understood only as the direction towards which the body moves, he had already made this claim in his first letter to Mersenne in 1637, and strongly criticizes Descartes' and Clerselier's mixing of the *determinatio* (of the movement) and velocity and force (of this movement), which may be considered as being completely independent in the horizontal *determinatio*, but having this kind of relation (which is nevertheless not explained clearly) in the vertical *determinatio*.

Even after twenty years of the publication of the *Dioptrique* nothing new has been proposed, as an objection or as a contribution to reject or to support Descartes' law of refraction, the same initial confusion remains, now becoming a double accusation: there is confusion on both sides, and it is because of this confusion that Fermat considered that the law of refraction was not acceptable since it had not yet been proved; and, according to Clerselier, it is this confusion that prevents his correct understanding of the law. We believe that it is this that led Fermat to work on a proof (or proofs) for the law of refraction.

Fermat's theory of refraction

As we mentioned in the Introduction, the second period in which Fermat worked on the law of refraction extends from 1657 to 1664. And it is during this period that Fermat writes two proofs of the law. These proofs can be difficult to follow because the author frequently intermingles elements of his anti-Cartesian criticism, particularly in his synthetic proof. Another preliminary remark worth making is that Fermat changed his mind as to the ease of his analytical proof. When he reduced the optical question to a question of geometry in 1637, he thought that, due to his method of *maximis* and *minimis*, the proof would be simple, but later on it would prove to be full of difficulties. Writing to Cureau de la Chambre in August 1657, he claims that « la chose est aisée et qu'un peu de géométrie pourra nous tirer d'affaire » (Fermat 1891-1912, 2: 355); and even further, he claims « Je vous garantis que j'en ferai la solution quand il vous plaira et que j'en tirerai même des conséquences qui établiront solidement la vérité de mon opinion » (Fermat 1891-1912, 2: 358-359). But it is four years later, on January 1st 1662, that he reports to the same correspondent in a letter which he adjoins to his text on the analytic proof of refraction: « (il) m'a été nécessaire d'avoir en cette occasion recours à ma méthode de *maximis et minimis*, qui expédie cette sorte de question avec assez de succès » (Fermat 1891-1912, 2: 460).

Let us examine these two letters to de la Chambre as a testimony of two moments of the genesis and culmination of the analytic and synthetic proofs. In August 1657 Fermat seems to have just finished reading de la Chambre's treatise *La Lumière* (de la Chambre 1657), which had recently been published, and as a breath of fresh air he accepts the principle through which the author explained the principle of reflection: that nature always acts through the shortest and easiest means possible (PS from now on). Using this principle de la Chambre had explained the equality of the two angles of incidence and reflection.

Fermat's first reaction was to propose a way to extend PS in order to explain the nature (and cause) of refraction of a ray of light. His idea is explained to de la Chambre as follows: based on the same diagram that de la Chambre conceived to reject the utilisation of PS for the refraction of light, Fermat observes that there is one point missing in the argument given by de la Chambre.

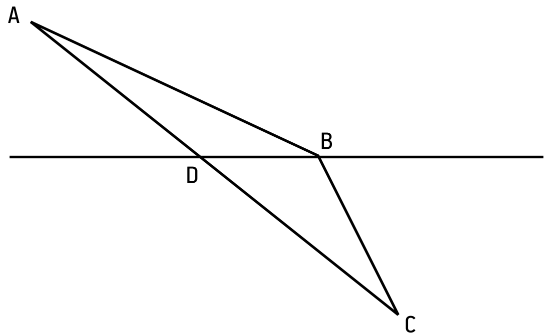


Figure 2 : *Fermat 1891-1912, 2: 356*

If a light ray CB is refracted to the ray BA when changing from a medium C to a medium A, assuming always that it is easier for light to move in a rare medium than in a dense one, contrary to Descartes' assumption, PS would lead one to conclude that light should move in a straight line, since (the sum of the) lines CD+DA is less than CB+BA. This being true does not disqualify PS as a means to explain refraction since it becomes necessary to consider the *resistance* to the passage of light of the medium (greater in a dense medium, lighter in a rare one). If the resistance which the medium opposes to the light ray, or the resistance against which light fights to go through a medium, does not change, that is if the line DB does not divide two different media, the ratio of the resistances is the same as the ratio of the lines. But if the resistances are different as the media are different, then the ratio $R_C : R_A$ is equal to the ratio CB : m, where m is a line (whose length is) different to the line DB. « Comme : si la résistance par le milieu A est double de la résistance par le milieu C, la résistance par CB sera à la résistance par BA comme la ligne CB au double de la ligne BA ; et si la résistance par le milieu C est double de la résistance par le milieu A, la résistance par CB sera à la résistance par BA comme la ligne CB à la moitié de la ligne BA » (Fermat 1891-1912, 2: 357). Now the problem seems clear to Fermat: it is to find the point B where the sum of these lengths, the line CB and double the line BA, or the line CB and half the line BA, would become, according to PS the minimum length. And Fermat concludes, « si nous supposons que la chose est déjà faite» (Fermat 1891-1912, 2: 358) the problem becomes the following geometric problem:

> «Étant donnés les deux points C et A et la droite DB, trouver un point dans la droite DB auquel si vous conduisez les droites CB et BA, la somme de CB et de la moitié BA contienne la moindre de toutes les sommes pareillement prises, ou bien que la somme de CB et du double de BA contienne la moindre de toutes les sommes pareillement prises». (Fermat 1891-1912, 2: 358)

We should underline the fact that the diagram in Fig. 2 appears to be the one where Fermat tries to explain for the first time his own view point on what refraction is and how it is to be understood. The problem then becomes the following: to find the point B on the line dividing two different media where a light ray from a point C in the first medium is refracted to a point A in the second medium. It is to this problem that Fermat assures de la Chambre he will obtain a solution without much trouble, though it will be a modified version of this (geometric) problem which he will solve four years later using his method of maxima and minima.

As we have explained, it is this letter which de la Chambre passes on to Clerselier and which starts a new period of discussion between Fermat and the Cartesians, now represented by Clerselier, and whose arguments in favour and against Descartes' proof of the law of refraction have been analysed above. It is worthwhile to mention that in the letter sent to Clerselier on June 16, 1658, amidst the new period of dispute, Fermat not only rejects Descartes' explanation of the refraction of light using a mechanical image of a moving body but challenges Clerselier to propose another acceptable explanation improving the one he has taken from de la Chambre: « Je réplique que le mouvement de la balle et la refraction ne se ressemblent que par la comparaison imaginaire de M. Descartes [...] (et) de quelque biais que vous le preniez, il faudra examiner les principes secrets dont sert la nature en produisant la réfraction, et si celui que j'ai touché dans ma lettre à M. de la Chambre ne vous plait pas, je souhaite qu'il vous en vienne de meilleurs dans l'esprit, et que cette vieille dispute aboutisse enfin à la pleine et entière découverte de la vérité» (Fermat 1891-1912, 2: 412).

Now, in January 1662, Fermat sends a new letter to de la Chambre, and includes a new argument which would turn out to be the key for his analytic and synthetic proofs. Instead of taking into account the distances traversed by light it is now time which is considered, he does this in a way that is consistent with his conviction that time should be shorter in a rare medium than in a denser one. Then if the lines AD and DC are equal (say that D is the centre of a circle through A and C), and assuming that the time $t_{AD}<t_{DC}$ since the medium A is rare and the medium C is denser, (and in fact $t_{AD}=kt_{DC}$) then the measure of this movement (which is now accepted as a movement) could be represented by the sum kAD+DC (usually if C is a dense medium $k < 1$). If the problem is to determine the point B where the light ray is refracted from the point A to the point C, then kAB+BC is a minimum length since it comes from the shortest time.

It is assuming that such a point has been found, and that time is minimum, that the ratio of the sines of the angles is proved to be constant, leading to Descartes' proportion and, according to Fermat, giving an end to his old discussion: Descartes' proportion (that is, the law of refraction) is correct, even though the proof given by Descartes is not, and could never be recognized as such. In other words, Fermat provides the correct proof, confirming Descartes claim and disqualifying his own ancient doubts about this law:

> « Après avoir couru par toutes les équations, multiplications, antithèses et autres opérations de ma méthode, et avoir enfin conclu le problème que vous verrez dans un feuillet séparé, j'ai trouvé que mon principe donnait justement et précisément la même proportion des réfractions que M. Descartes a établie». (Fermat 1891-1912, 2: 461-462).

After this expression of truce and a desire of peace Fermat claims « que devons nous conclure de tout ceci? Ne suffiront-il pas, Monsieur, aux amis de Descartes que je lui laisse la possession libre de son théorème ? N'aura-t-il pas assez de gloire d'avoir connu les démarches de la nature dans la première vue sans l'aide d'aucune démonstration ? Je lui cède donc la victoire et le champ de bataille » (Fermat 1891-1912, 2: 462). Afterwards, in a touching post-scriptum to this letter, Fermat states that he has not included all the details, since the long and clear explanation of his method (of maxima and minima) can be found in Herigon's *Cours Mathématique*, but, if de la Chambre considers it necessary, a synthetic proof could be added to the analytic one: « si vous m'ordonnez de parcourir tous les détours de l'analyse en forme, je le ferai et je n'aurai pas même beaucoup de peine à faire la démonstration par la composition, c'est-à-dire en parlant le langage d'Euclide » (Fermat 1891-1912, 2: 463).

Before we analyse Fermat's analytic and synthetic proofs, it is worthwhile to notice that the analytic proof proves that if time is minimum then the sine ratio is constant whereas the synthetic proof proves that if the sine ratio is constant, then time is minimum. It is therefore important to note that we are not dealing with two different methods for the same result, but indeed two different (and complementary) results.

The Analytic Proof

We think that there are four reasons supporting the idea that the proof given in *Analyse pour les refractions* can be considered as *analytic*. The first is that it is a method of discovery: the discovery of the right ratio regulating the correspondence of angles of incidence and refraction. The second reason is that it depends on the *maximis* and *minimis* method often called « mon analyse» by Fermat. The third reason is that it supposes the problem to be solved, insofar as one considers the points of departure, arrival and crossing from one medium to another of the ray as known. The final reason is that the proof goes from a principle to the effects, the principle being that of minimum time and the effect being the ratio of the sines. This last point is rather curious as a characteristic of an analytical approach since it is quite common to see it as a watermark of the synthetic way.[11]

11 Technically, Fermat's analytical method resembles Leibniz's proof as presented in the *Nova methodus pro minimis et maximis* (Leibniz 1684), with two differences. The first concerns the quantity that must be minimal; in Leibniz, it will be the difficulty of the trajectory and for Fermat, time. However, the two coincide if we consider that the characteristic

Analyse pour les refractions

Fermat begins this text by considering a circle as seen in Figure 3 where AFDB is the diameter and separates two distinct media. The medium on the side of ACB is taken to be less dense than the medium on the side of AIB.

Let D be the centre of the circle and CD a radius; Fermat defines the problem as finding the radius DI (or the point I) upon which the refracted ray will fall. To do this he considers the perpendiculars to the diameter, CF and IH and details the fact that point C, the diameter AB, point D and point F (and hence FD) are given.

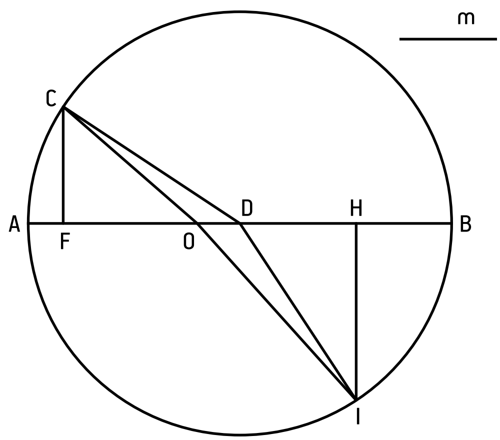


Figure 3: *Fermat 1891-1912, 3: 150*

Fermat then supposes that the ratio of the resistance of the denser medium (R_{AIB}) to the resistance of the less dense medium (R_{ACB}) is the same as the ratio of FD to another given line m (given outside the figure). It should be the case that m < FD by an axiom that Fermat considers « plus que naturel» (Fermat 1891-1912, 3: 150); that is that the resistance in the less dense medium is less than the resistance in the denser medium.

Fermat then states that what needs to be done is to measure, by means of the straight lines m and FD, the motion along the lines CD and DI. It is also possible to represent motion along these two lines, the incident and the refracted rays, by the sum of the two products: CD x m + DI x DF. The question is then reduced to finding a point H on AB such that when the line HI is drawn perpendicular to the diameter AB, and D and I are joined, then the area CD x m + DI x DF is minimal. It is clear that a slight but important change has been

index of the medium is inversely proportional to the speed of light within the medium. The second difference is the power of Leibniz's differential algorithm which solves in a few lines the problem posed by the calculation.

introduced by Fermat in his analytic proof, compared with the brief explanation given to de la Chambre in his letter of August 1657. As we said above, in his letter Fermat claims that if a light ray is refracted when passing from a rare medium to a denser one, and both the source and end points are known, the problem is to find the position of the point on the dividing line where the ray is refracted. Now, in his analytic proof, Fermat assumes that the sought point on the dividing line is the point D, the centre of the circle; hence the problem is not to determine the position of this point but to determine the position of the point I reached by the refracted ray below the dividing line on the circle.

Once that Fermat has set out the goal of the text, he proceeds to solve the problem; he claims he will do so by using « notre méthode, déjà répandue parmi les géométres et exposée depuis environ vingt ans par Hérigone dans son *Cursus mathematicus*» (Fermat 1891-1912, 3: 150). Let CD = DI = n, DF = b, hence CD x m + DI x DF = nm + nb, and let DH, the unknown quantity which is treated as if it were known, be equal to (the unknown) a.

According to the *maximis* and *minimis* method it is required that nm + nb be a minimum, so let e be another unknown quantity with which an arbitrary straight line DO is set («soit, pour l'inconnue e, une droite arbitraire DO», (Fermat 1891-1912, 3: 150)) and join CO and OI. Then we have on one hand that $CO^2 = n^2 + e^2 - 2be$, and on the other hand that $OI^2 = n^2 + e^2 + 2ae$, hence $CO = \sqrt{n^2 + e^2 - 2be}$ and $OI = \sqrt{n^2 + e^2 - 2ae}$. At this point, Fermat states that the sum CO x m + OI x b must be «adégalée», according to the art, to the sum CD x m = DI x b, but since $CO \times m = \sqrt{m^2n^2 + m^2e^2 - 2m^2be}$ and $OI \times b = \sqrt{b^2n^2 + b^2e^2 + 2b^2ae}$, this means that $\sqrt{m^2n^2 + m^2e^2 - 2m^2be} + \sqrt{b^2n^2 + b^2e^2 + 2b^2ae}$ must be adequated to nm + nb.

From this it would be possible to determine, under the condition that nm + nb be a minimum, the value of the unknown quantity DH = a. Let us remark that in this «adequation» the left-hand side includes two indeterminate quantities, a and e, while the right-hand side contains only «constant» (known) quantities. Considered as mere quantities, the line segments CD and CI are both radii of the circle which are assumed to be equal to n; the same happens with the (given) line FD which is assumed to be equal to the (known) quantity b, since the point F is given as the point C is given. The same occurs with the quantity m which is *given* as the fourth proportional: $\frac{R_{AIB}}{R_{ACB}} = \frac{FD}{m}$. In fact, when the problem is stated as the area CD x m + DI x DFCD x m + DI xDF becoming minimal, it is necessary to analyse the equation/adequation together with the diagram involved (Figure 3) because what Fermat's method of *maximis* and *minimis* allows us to determine is the unknown quantity a = DH which in fact provides the value of the abscissa of the point I whose *position* is to be determined.

To do this Fermat proceeds to eliminate the square roots by squaring both sides of the equations twice and divides through by e. This takes him to the following equation:

$4b^2m^2e^3 - 8b^3m^2e^2 + 8ab^2m^2e^2 - 16ab^3m^2e - 8b^3m^2n^2 + 8b^3m^2n^2e + 8ab^2m^2n^2 =$
$4a^2b^4e + 4ab^4e^2 - 8ab^3m^2e - 8ab^3mn^2 + 4ab^2m^2e^2 + e^3b^4 - 4b^3m^2e^2 - 4b^3mn^2e +$
$4b^2m^4e + 8b^2m^3n^2 + 2b^2m^2e - 4bm^4e^2 - 4bm^3n^2e + e^3m^4$

After equating coefficients for $e^3 e^3$, $e^2 e^2$, $e^1 e^1$ and constants (with regard to e) we arrive to the equality $m^2 = b^2$ and from this equality it can be deduced (again by equating coefficients) that $a - m = 0$. This is the equation Fermat wanted to obtain: « on arrivera [...] à l'équation la plus simple possible entre a et m, c'est-à-dire qu'après avoir fait disparaitre les obstacles opposes par les radicaux, on trouvera que la droite DH de la figure est égale à la droite m» (Fermat 1891-1912, 3: 151).

To obtain the position of the point I where the refracted ray reaches the circle below the dividing line, given the straight lines CD and CF, Fermat considers lines DF and DH in the ratio of the resistance in the denser medium to that of the resistance in the less dense medium, in the ratio of b to m. Then by drawing the line HI, perpendicular to the diameter AB, I is the point where this perpendicular line crosses the circle, it is clear that the refracted ray will cross the circle at I as the ray passes from a rarer medium to a denser one and hence bends on the side of the perpendicular. To conclude, Fermat then states that this result agrees with the theorem discovered by Descartes and that the analysis he has carried out, starting from his principle, gives this theorem a rigorously exact proof: since his proof leads to the value a = DH = m, the ratio of the resistances $\frac{R_{AIB}}{R_{ACB}}$ becomes equal to the ratio $\frac{FD}{DH}$ which is the ratio of the sines of the incident and the refracted rays.

The Synthetic Proof

As we quoted above, in the same post-scriptum to the letter sent to de la Chambre in January 1662, Fermat claims that another proof of the refraction law is also possible, a proof *by composition*, according to Euclid's language. This *synthetic proof* is given in the brief text *Synthèse pour les refractions* (Fermat 1891-1912, 3: 151-156) presumably accompanying the previous *Analyse pour les refractions*. In the synthetic proof only the ancient geometric language is used; no place is given for any "analytic" or "algebraic" operation, and, in Euclid's language only proportions between the geometric magnitudes are considered.

The *synthetic* proof does not present a method of discovering the law of refraction since this law serves as a starting point. It is rather a formal attempt to prove the principle of minimum time, however, there is something interesting as this principle is not of a geometrical nature and it is absolutely certain in Fermat's eyes and, according to him, in any reasonable mind. In fact, this synthetic proof does not establish anything new since it gives a hypothetical status to the ratio of the sines and it proves a relation (the minimum time) which is *a priori* certain. This, however, does not make the proof uninteresting, it aims to establish that the law of sines and the rule of minimum time are equivalent. One of the consequences of this is that the Cartesian proof leads to the rule of minimum time (or rather, in the Cartesian lexicon, the minimal difficulty) and, to our knowledge, none of the Cartesians involved in this debate make this observation.[12]

Synthèse pour les refractions

Fermat begins his text by commenting on Descartes' law of refraction, he states that Descartes' law conforms to experience but that to prove it Descartes has had to use a postulate that appears to be contrary to the nature of light. Descartes, as we have mentioned above, relies on the hypothesis that light moves easily and faster in a denser medium than in a rarer one. Fermat believes that if this is taken as a starting point then the proof of any law must be erroneous. He claims that to state the true law of refraction he has used the opposite principle, that is, that light moves easier and faster in less dense media. Nevertheless, even though Fermat relies on the opposite principle he arrives at the same law of refraction and he questions this point: « Est-il possible d'arriver sans paralogisme à une même vérité para deux voies absolument opposées? » (Fermat 1891-1912, 3: 152).

Fermat's synthetic proof rests on the same postulate PS: « la nature opère par le moyens et les voies les plus faciles et les plus aisées » (Fermat 1891-1912, 3: 152), and he comments that it is important to make this distinction and not just say that nature acts along the shortest path and to back up his point he compares this to Galileo's work where it is not the shortest line or space that is considered but those that are more easily covered. Once this has been set out Fermat begins his proof. He considers two different media and takes a circle AHBM whose diameter ANB separates two media (N is taken as the centre of the circle). He supposes one medium to be denser than the other and takes the bottom half of the circle to be denser (see Fig. 4). This means that the velocity of the moving object along MN will be greater than the velocity of the same object along NH.

12 If one wishes to compare this situation with a very classical piece of mathematics concerning two demonstrations –one which is analytic and one which is synthetic– for the same problem, we can consider the two Archimedean methods for solving the quadrature of the parabolic segment. Two of the reasons for qualifying the first one as analytic are the same as Fermat's: first, its status as a method of discovery (the proportion $\frac{4}{3}$ in Archimedes), secondly, the use of infinitesimal methods in the calculations (indivisibles in Archimedes). However, the comparison stops here since the synthetic method in Archimedes does not change object: it must establish that the proportion of the two areas is $\frac{4}{3}$. Moreover, there is no principle-effect that is comparable to Fermat's, unless we want to consider the pair principle of the excluded middle - ratio $\frac{4}{3}$. However, even then, we would have a difficulty because the two Archimedean proofs lead to the same statement: the ratio is $\frac{4}{3}$, whereas in Fermat, the first proof leads to the statement "the ratio of the sines is constant" and the second leads to the statement "travel time is minimum".

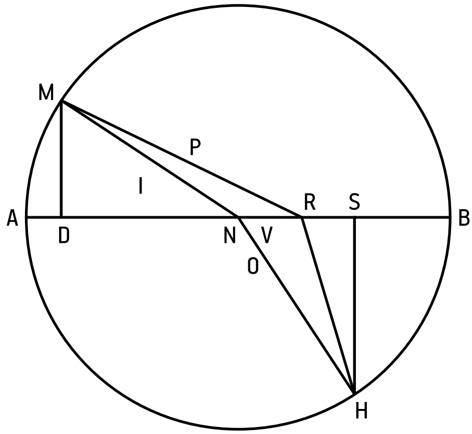


Figure 4: *Fermat 1891-1912, 3: 152*

In other words, if we denote any segment by its endpoints, say A and B, and the velocity of an object moving along the segment AB by V_{AB}, then in this case we have that V_{MN} is greater than V_{NH}. Fermat takes two lines going from M to H, MNH and MRH with the points N and R on the diameter and supposes that the object moves uniformly in both media. This implies that the ratio of the time of motion along MN to the time of motion along NH will be the product (*the compounded ratio*) of the ratio of MN to NH with the inverse ratio of the velocities: $\frac{T_{MN}}{T_{NH}}=\frac{MN}{NH}\frac{V_{NH}}{V_{MN}}$, where T_{MN} denotes the time taken to get from M to N.

He then takes a point I on MN such that the velocity on MN is to the velocity on NH as MN is to NI, and since $V_{MN} > V_{NH}$ it follows that the point I must be placed between M and N. In other words, $\frac{V_{MN}}{V_{NH}}=\frac{MN}{IN}$ and $\frac{T_{MN}}{T_{NH}}=\frac{IN}{NH}$. In the same manner, Fermat obtains a point P between M and R such that, $\frac{V_{MR}}{V_{RH}}=\frac{MR}{RP}$ hence $\frac{T_{MR}}{T_{RH}}=\frac{PR}{RH}$. From this Fermat claims that $\frac{T_{MNH}}{T_{MRH}}=\frac{IN+NH}{PR+RH}$ and indeed this is the case if one uses the fact that the velocity in the same medium is the same regardless of the path and observes that $\frac{T_{MN}+T_{NH}}{T_{NH}}=\frac{IN+NH}{NH}$ and $\frac{T_{MR}+T_{RH}}{T_{RH}}=\frac{PR+RH}{RH}$.

Now Fermat explains his goal in the proof and how he plans on achieving said goal:

> «Or, puisque c'est la nature qui dirige la lumière du point M vers le point H, nous devons chercher un point, soit N, par lequel la lumière, en s'infléchissant ou se réfractant, parviendra dans le temps le plus court du point M au point H ; car on doit admettre que la nature, qui mène le plus vite possible ses opérations, visera d'elle-même ce point-là. Si donc la somme IN + NH, qui mesure le temps du mouvement sur la ligne brisée MNH, est une quantité minima, nous aurons attient notre but». (Fermat 1891-1912, 3: 153)

Fermat says that Descartes' statement of the theorem gives this minimum but that he will prove this fact by means of a true geometric argument, and clearly provides the statement of the theorem:

«Si du point M on mène le rayon MN , que du même point M on abaisse la perpendiculaire MD, puis que l'on prenne $\frac{DN}{NS}$ dans le rapport de la plus grande vitesse à la moindre, qu'enfin on élève en S la perpendiculaire SH et que l'on mène le rayon NH, la lumière incidente au point N dans le milieu rare se réfractera dans le milieu dense du côté de la perpendiculaire vers le point H». (Fermat 1891-1912, 3: 153)

Fermat claims that it is this statement that is in agreement with our Geometry and now that the precise geometric goal has been set out he continues as follows: in Figure 4, he assumes that N is the centre and draws MD perpendicular to AB and lets the point S be set out so that the ratio $\frac{DN}{NS}$ be equal to the ratio $\frac{V_{MN}}{V_{NH}}$, hence DN > NS. From the point S a perpendicular SH to the diameter is drawn and H is the point of intersection of this perpendicular with the circle; now a point I is taken on the line MN so that $\frac{DN}{NS}=\frac{MN}{NI}$. Fermat claims that IN + NH is minimum, that is, that if any other point R is taken on the radius NB, and MR and RH are drawn, and again a point P on the line MR is taken so that $\frac{DN}{NS}=\frac{MR}{RP}$ then PR + RH > IN + NH

To achieve this, Fermat considers the perpendicular RO to NH, so that $\frac{MN}{ND}=\frac{RN}{NO}$; a point V on the line NH is also taken so that $\frac{DN}{NS}=\frac{NO}{NV}$. It is clear that since DN < MN, then NO <NR, NS <ND, and NV < NO. Fermat then claims that "by Euclid" $(MR)^2 = (MN)^2 + (NR)^2 + 2(DN \times NR)$.[13] But since the proportion $\frac{MN}{ND}=\frac{RN}{NO}$ holds, it follows that MN x NO = DN x NR, hence $(MR)^2 = (MN)^2 + (NR)^2 + 2(MN \times NO)$. However, since NR > NO then $(NR)^2 > (NO)^2$, hence $(MR)^2 > (MN)^2 + (NO)^2 + 2(MN \times NO) = (MN + NO)^2$. In other words, Fermat obtains the inequality MR > MN + NO.

On the other hand, the points I, O and V have been set out so that $\frac{DN}{NS}=\frac{MN}{NI}=\frac{NO}{NV}$, therefore $\frac{DN}{NS}=\frac{MN+NO}{NI+NV}$. However, the point R has also been set out so that $\frac{DN}{NS}=\frac{MR}{RP}$, hence $\frac{MN+NO}{NI+NV}=\frac{MR}{RP}$. But from the inequality MR > MN + NO that Fermat had just proved, it follows that PR > IN + NV. Since the goal is to prove that PR + RH > IN + NH, it now suffices for Fermat to prove that RH > NV. To do this Fermat observes that "by Euclid", in triangle NHR one has that $(RH)^2 = (NH)^2 + (NR)^2 - 2(SN \times NR)$ but, by construction $\frac{NH}{DN}=\frac{NR}{NO}$ and $\frac{DN}{NS}=\frac{NO}{NV}$ so, *ex æquo*, $\frac{NH}{DN}\frac{DN}{NS}=\frac{NR}{NO}\frac{NO}{NV}$. This in turn implies that $(RH)^2 = (HN)^2 + (NR)^2 - 2(HN \times NV)$, and as $(NR)^2 > (NV)^2$ it follows that $(RH)^2 = (HN)^2 + (NV)^2 - 2(HN \times NV)$. Fermat then claims that, again

13 We would like to point out here that we follow notation as it appears in the edition we have used of Fermat's *Œuvres*, though we have no direct evidence of the notation used in his synthetic proof. However, since he claims that this proof is made "en parlant le langage d'Euclide", which means that the equalities among plane (rectilinear) figures and the proportions of their sides are the means through which geometric synthetic proofs are made, and since Fermat is also considering equalities between plane (geometric) figures, this equality should be understood as Q[MR] = Q[MN] + Q[NR] + 2R[DN,NR]. We will use the usual notation $(MR)^2$ for Q[MR],... but we should not forget the fact that throughout the synthetic proof Fermat is dealing with plane figures.

by Euclid, $(HV)^2 = (HN)^2 + (NV)^2 - 2(HN \times NV)$, hence $(HR)^2 > (HV)^2$ which implies HR > HV which is exactly what was needed to be proved. Fermat completes his proof by showing that if another point R' is taken on the radius AN it is also the case that PR' + R'H > IN + NH. He concludes that «Il est donc certain que la somme des deux droits PR, RH, quand même elles ne formeraient qu'une droite unique PRH, est toujours supérieure à la somme IN + NH» (Fermat 1891-1912, 3: 156). Therefore Fermat's synthetic proof shows that when the point S is determined, using $\frac{V_{MN}}{V_{NH}}=\frac{DN}{NS}$, and a point I is taken on the radius MN so that $\frac{DN}{NS}=\frac{HN}{NI}$, then the sum IN + NH is a minimum. In this case that nature acts by the easiest means according to PS, is that it follows the shortest distances only when this is done also with the least time, since when this is not the case, nature follows not the shortest path but that which requires the least time.

Analysis and Synthesis

There are of course some important differences between the analytic and the synthetic proofs when they are compared, but there are also some common aspects which are interesting in addition to what we have already underlined. It appears that in both proofs Fermat assumed the same principle PS, and in particular, in the analytic proof he uses it to claim two things:

The ratio of the resistance of the denser medium to the rarer medium $\frac{R_{AIB}}{R_{ACB}}$ is the same as the ratio $\frac{FD}{m}$ (Figure 3), and that m < FD by an axiom considered «plus que naturel».

By means of the lines FD and m the movement of light along the lines CD and DI is measured, that is, if the movement through one media is represented as the product of the distance multiplied by the resistance, the movement along the line CD being represented by the product $CD \times R_{ACB}$ and the movement along DI by the product $DI \times R_{AIB}$, the movement as a whole is represented, assuming the proportion $\frac{R_{AIB}}{R_{ACB}} = \frac{FD}{m}$, by the sum of the two products: CD x m + DI x DF.

Fermat then concludes that in order to find the point I where the refracted ray arrives, it is first required to find the point H on the dividing line AB from which the perpendicular line HI is drawn, the refracted ray is obtained when D and I are joined; but this task is achieved when the area CD x m + DI x DF is *minimal*.

On the other hand, the synthetic proof which is guided by the interpretation of PS we gave above shows that if the light ray emerges from a point M in the rare medium AMB and on the circle whose centre is N, and crosses the line AB to a denser medium AHB, then it arrives at a point H (on the lower part of the circle) which must be determined. If the ratio of the velocities in these two media, which is the same as the inverse ratio of the times employed to traverse them, is represented as the ratio of two lines, $\frac{V_{MN}}{V_{NH}} = \frac{T_{NH}}{T_{MN}} \frac{HN}{NI}$, where I is a point on the line MN since $V_{MN} > V_{NH}$, and a point S is taken on the radius NB such that $\frac{DN}{NS}=\frac{V_{MN}}{V_{NH}}$, hence $\frac{DN}{NS}=\frac{HN}{NI}$, then Fermat claims that if the perpendicular line SH to the diameter AB is drawn and H is the point of intersection of this perpendicular with the circle, H is the point which was to be determined. This means that if the sum of the lines IN + NH measures the time of the movement along the broken line MNH, then it follows from PS that the sum IN + NH is a *minimum*.

It is clear that when we try to unify the two diagrams involved in Figures 3 and 4, the line FD in the first one is the same as the line DN in the second, while the line m in the first one, which is not identified with any segment is proved to be equal to the line segment DH, which appears to be the segment NS in the second. Therefore, the ratio $\frac{DN}{NS}$ in the second diagram is the same as the ratio $\frac{FD}{m}$ in the first one, but this means that the ratio of the resistance of the denser medium to the rarer medium, $\frac{R_{AIB}}{R_{ACB}}$, should be the same as the ratio of the velocity of the light through the less dense medium to the velocity through the denser medium $\frac{V_{MN}}{V_{NH}}$. Hence it should be emphasised that the ratios which are considered by Fermat as given, that is, the ratio $\frac{R_{AIB}}{R_{ACB}}$ of the resistances involved in the analytic proof, and the ratio $\frac{V_{MN}}{V_{NH}}$ of the velocities involved in the synthetic proof, are equated to a ratio of two lines, lines which appear to be the sines of the angles of incidence and of refraction.

Fermat works in apparently the same manner in both proofs, that is, in both proofs he assumes that the point of refraction is the centre of the circle and the light rays are radii., He then proves that the shortest and easiest means possible through which nature acts, according to PS, is precisely when the light ray goes through the centre of the circle. Hence it is an argument of minimality which supports Fermat's proof, not an idea claiming that the lines involved (FD and DH, or DN and NS) are conceived as being *components* of the movement of light.

Conclusion

In addition to the four reasons we mentioned above justifying the *analytic character* of Fermat's *analytic proof*, some remarks are necessary to fully understand what is at stake with both, the analytic and the synthetic proofs, but also to understand what is at stake when they are proposed to solve a physical problem.

We know that Fermat's first contact with mathematics came from Viète's works. It was when reading the *Isagoge* that he first became aware of and read Apollonius and Pappus, and even though it is clearly not our aim to reconstruct Fermat's mathematical career, we must acknowledge that his first mathematical texts deal with some of the texts constituting the Domain of Analysis (*topos analyomenos*) as described by Pappus in Book VII of his *Mathematical Collection*. It was around the years 1636-37 that Fermat wrote his reconstruction of the two Books of Apollonius's *Plane Loci*, and he later wrote on the Book of *Contacts* and on Euclid's Books of *Porisms*. In fact, we could say that his method of maxima and minima was conceived, exactly as Descartes conceived his (algebraic) method, as a modern device into which the ancient geometric analysis ought to be translated into. Fermat and Descartes appear in this way as two mathematicians who, each one in his own way, contributed to the reconstruction of geometric analysis, a task first conceived by Viète, after his own reading of Commandino's latin translation of Pappus' *Mathematical Collection*.

Whether his method of maxima and minima satisfied Fermat completely or not when he conceived it as his own contribution (say methodological contribution) to the constitution of "modern analysis", it is clear that no matter how broad its scope was by the end of 1630's, we may be sure that he did not contemplate the fact that it could be useful to understand physical phenomena. Let us recall that Fermat constantly declared he was not a physicist, hence his method was certainly not conceived to be used to understand the reflection and refraction of light. As we claimed above, the first group of letters on this topic written around 1637-38 focused on criticising Descartes' method used in the *Dioptrique* –which could hardly be considered as being *analytic* in the sense that his method for solving the key problems in his *Géométrie* is considered analytic – but without proposing any original idea of how refraction could and should be studied.[14]

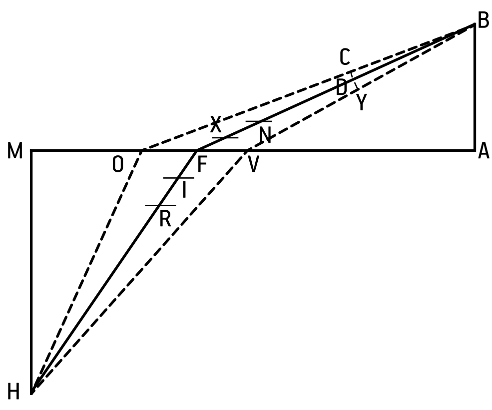
But we could say something more about Fermat's analytic method of maxima and minima for proving the law of refraction by comparing his two letters sent to Cureau de la Chambre in August 1657 and in January 1662. Certainly, as it was explained to de la Chambre in his first letter, the geometric problem translating the physical question becomes a geometric problem as conceived by the ancients: given two points A and C, one at each side of a given line DB, the problem is to find the position of the point B on this (given) line such that a (given) multiple of the line AB together with the line BC is a minimum (Fig. 2). Of course, Fermat could claim that the novelty in relation to the ancient's geometric analysis is precisely the nature of the problem, to determine the position of a point B on a given line, in this sense it is a *locus problem*, but submitted to a *minimal condition*. But when we compare it with Descartes' analytic method developed in *la Géométrie*, it appears clearly that the difficulty that Fermat faced, and which explains the four years period between the formulation of this geometric problem and its solution, is that a frame of reference becomes necessary in order to find the position of the point B.

Four years later Fermat introduced this frame of reference through an ingenious coordinate system conceived to present a new version of this *locus-minimal* problem. First an impor-

14 On the other hand, we know that Descartes was not convinced that Fermat's method of maxima and minima could go as far as his own method could for the calculus of tangents. On this point, it is important to see Descartes' criticisms to Fermat's method (See the correspondence from Descartes to Mersenne, January 1638 (Descartes 1897-1913, 2).

tant change is introduced, as we already pointed out: in his *Analyse pour les refractions* the point of refraction is no longer the one which is sought, but is considered as given; hence the problem is no longer to find, on a given line representing the division of two medias, this point of incidence and refraction, but to find (Fig. 3), when the initial point C where the ray emerges and the point D on the dividing line where this ray is refracted are given, the point I (in a circle whose centre is D and radius equal to the line DC) from which the refracted ray DI is found. We could translate Fermat's analytic method in the following terms which clearly show his frame of reference as a coordinate system: since the position of the point C is given, we may assume that its coordinates are given by the lines FD and FC, and since the line CD is also given (in magnitude), the problem is to find the abscissa DH from which the orthogonal ordinate HI is found, providing the position of the point I and thus the direction of the refracted ray through the line DI. The maxima and minima method becomes the key to formulate the equation and to find the value of DH from it.

But this change and the (implicit) coordinate system which is thence introduced has for Fermat another important role to play, as he explains in 1664 to M. de ****, since it is *the nature* of the problem (to put it in terms of the ancient geometers) which is at stake:



« Lorsque les deux points B et F sont donnés, ou bien H et F, on peut trouver aisément le problème par les plans ; mais, lorsqu'on donne deux points, comme B et H, et qu'on veut chercher par eux le point de réfraction dans la ligne ou plan qui sépare les des milieux, en ce cas le problème est solide, et ne se peut construire qu'en y employant des paraboles, des hyperboles ou des ellipses». (Fermat 1891-1912, 2: 495)

Figure 5: *Fermat 1891-1912, 2: 489*

This means that the original solution which was proposed to de la Chambre in August 1657: to find (the position of) the point F on the line separating the two media where the light ray is refracted when coming from the point B, above the dividing line, and arriving to the point H below this same line, appears to be obtained only as a solution to a *solid problem.* This difficulty is somehow explained to de la Chambre already in his letter of January 1662, where Fermat explains that the solution of the original problem as formulated in August 1657, to determine the position of the point of refraction, the point F in Figure 5, became a problem involving a long and difficult calculation hence it appeared not to be the easiest one. Contrary to this, the solution given in January 1662 to de la Chambre, which depends on the introduction of the frame of reference as we have explained, reveals to be, when the point of refraction is now considered to be given and the problem is to find the position of the point H in Figures 4 and 5 (the point I in Figure 3), a *plane problem*.

It should be noted that besides the different alphabet symbols employed in Figures 3, 4 and 5, they all represent essentially the same phenomenon, so in the three of them a point above the line dividing two different media is conceived as s source of a light ray. If we remain with the symbols in Figure 5 we could say that when the ray emerging from the point B is assumed to be refracted on a point F on the line MA, and the position of the two points is assumed to be given, then the determination of (the position of) the point H appears as the key task to achieve since though this point it is in fact the direction of the refracted ray FH which is obtained. Fermat's proof, obtained from the principle PS states in fact that under this principle, the ratio $\frac{AF}{FM}$ is constant, where the lines segments AF and FM represent the abscissa of the point A and H. Now Fermat proved that in order to reach the point H the ray emerging from the point A should refract precisely on the point F, the centre of a circle whose radius is n = FA = FH; that is according to PS any different path as BOH or BVH would take more time than that employed by the path through the centre F. Now we should point out that according to the law of refraction that Fermat has found, and which is the same as Descartes' law of refraction, if the point A is conceived as a source of light, and two rays are considered emerging from it, one of them being the ray AF and the other being AV, then the ray refracting on the point V does not take the direction VH but it takes a different path. A circle whose centre is the point V and through the point A is to be considered in order to determine the position of the point reached by this ray on the second circle. From the law of refraction just proved, the abscissa VP of this point on the second circle is obtained as the fourth proportional: $\frac{AF}{FM}=\frac{AV}{VP}$. Hence, we may conclude that Fermat's first act of acknowledgement of Descartes' law of refraction came from the acceptance of his diagram of a circle as the model where the light refraction phenomenon is well represented. Besides the negative opinion towards Descartes' interpretation about the way in which a ray of light crosses two different media, an opinion which never changed and made Fermat wonder about why the same (true) law could be found through different (and even opposite) interpretations, we think that a key role was played by the acceptance of Descartes' diagram. Whether the movement of a light ray is decomposed in two (orthogonal) movements according to the parallelogram law, or it is conceived as a natural movement searching the easiest (and faster) way, when the phenomenon is modelled as taking place in a circle whose centre is the refracting point, it appears clearly that the main assumption is that the same distance has been covered in the two different media.

These remarks could help to understand Fermat's difficulty faced after his letter to de la Chambre of August 1657, because as the problem was stated therein (Figure 3) the point A from which the light ray emerged and the point C where the refracted ray is assumed to arrive do not belong (it is not at all an assumed hypothesis) to a circle whose centre is the point B where the ray refracts. In other words, if returning to the diagram in Figure 5, if the intermediate point P is consider as the point reached by the refracted ray, the solid nature of the problem posed to de la Chambre then appears since only the points A and R are given, as well as the position of the dividing line: the problem is a plane problem if this dividing line is conceived as the diameter of a circle where the points A and R lay.

A final remark should be made concerning Descartes' and Fermat's solutions since they both arrive at the same law, although they each one take as starting point opposite hypothesis concerning the resistance of the medium to the transmission of the light ray. This leads to the paradoxical fact that from the same incident ray they obtain two different refracted rays: for Descartes, the refracted ray goes away from the normal (Figure 1), while for Fermat it comes closer to it (Figure 3). It is easy to see that the relation they both found leads to the same result because the ratio of the sines obtained by one of them is the inverse ratio of the one obtained by the other, but this ratio is equal to the ratio of the respective resistance of the respective media, which is explicitly conceived by them as being the inverse ratio of one another. On this point, it is again remarkable to read what Fermat wrote to M. de **** in 1664:

> « Il suit de là, qu'en posant mon principe, que la nature agit toujours par les voies les plus courtes, la supposition de M. Descartes est fausse, lorsqu'il dit que le mouvement de la lumière se fait plus aisément dans l'eau et les autres corps denses que dans l'air et les autres corps rares.

Car, si cette supposition de M. Descartes étoit vraie et que vous imaginiez qu'en ma figure l'air est du côté de H et l'eau du côté de B, il s'ensuivroit, en transposant la démonstration, que le rayon qui partiroit du point H et rencontreroit l'eau au point F, se romproit vers B, parce que, le mouvement par l'air étant plus lent selon la supposition de M. Descartes (...) la réfraction se feroit vers B, c'est-à-dire que le rayon s'ecarteroit de la perpendiculaire, ce qui est absurde et contre l'expérience ». (Fermat 1891-1912, 2: 495)

However, as it has been repeatedly claimed, both Descartes and Fermat, come to the same sine law of refraction; there is no doubt that an important lesson should be learned from the fact that despite their initial differences, their mathematical models, although different, provide the same answer which is satisfactory for both.

References

- Andersen, K. (1983), "The mathematical technique in Fermat's deduction of the law of refraction", *Historia Mathematica* 10(1): 48-62.
- Biard, J. , Rashed, R. (eds) (1998), *Descartes et le Moyen Âge*. Paris: Vrin.
- De la Chambre, M.C. (1657), *La Lumière*. Paris: P. Rocolet.
- Descartes, R. (1637), *La Dioptrique*, in *Discours de la méthode*. Leiden. Reprinted in Descartes 1897-1913, 4.
- Descartes, R. (1897-1913), *Œuvres de Descartes*. Edited by Adam, C., Tannery, P. 12 vols. Paris: Leopold Cerf.
- Fermat, P. (1891-1912), *Œuvres de Fermat*. Edited by Tannery, P., Henry, C. 4 vols. Paris: Gauthier-Villars et Fils.
- Giusti, E. (2009), "Les Méthodes des Maxima et Minima de Fermat", *Annales de la Faculté des Sciences de Toulouse* 18: 59-85.
- Jullien, V. (2006), *Philosophie Naturelle et Géométrie au XVIIe Siècle*. Sciences, techniques et civilisations du Moyen Âge à l'aube des lumières 9. Paris: Honoré Champion.
- Leibniz, G. (1684), "Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus", *Acta Eruditorum* 10: 467-473.
- Mahoney, M.S. (1973), *The Mathematical Career of Pierre de Fermat (1601-1665)*. Princeton, NJ: Princeton University Press.
- Pascal, B. (1964-1992), *Œuvres complètes*. Edited by Mesnard, J. 4 vols. Paris: Desclée de Brouwer.
- Sabra, A.I. (1981), *Theories of Light from Descartes to Newton*. Cambridge: Cambridge University Press.
- Strømholm, P. (1968), "Fermat's Methods of Maxima and Minima and of Tangents. A Reconstruction", *Archive for History of Exact Sciences* 5(1): 47-69.